

A BT-ALGEBRA OF TYPE B

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ABSTRACT. We introduce a bt-algebra of type B. As the original construction of the bt-algebra, we define this bt-algebra of type B by building from $Y_{d,n}^B$. Notably we find a basis for it, a faithful tensorial representation, and we prove that it supports a Markov trace, from which we derive invariants of classical links in the solid torus.

1. INTRODUCTION

The algebra of braids and ties, known as well as the bt-algebra, was defined originally by Aicardi and Juyumaya in [1], having as goal to construct new representations of the braid group. Later, it was observed that its generators and relations have a diagrammatical interpretation in terms of braids and ties, hence its name, see [2, Section 6]. For a positive integer n , the bt-algebra with parameter u is denoted by $\mathcal{E}_n(u)$, and its definition is obtained by considering abstractly as the subalgebra of the Yokonuma–Hecke algebra $Y_{d,n} := Y_{d,n}(u)$ generated by the braid generators and the family of idempotents that appear in the quadratic relations of these generators. Thus, there is a natural homomorphism from \mathcal{E}_n in $Y_{d,n}$, which is injective for $d \geq n$, see [7], cf. [2, Remark 3].

In [1] was proved that the bt-algebra has finite dimension, although that the authors couldn't get an explicit basis for it. This problem was solved later by Ryom-Hansen in [18], who constructed a basis for \mathcal{E}_n , proving that the dimension of the algebra is $b_n n!$, where b_n is the n -th bell number. He also constructed a faithful tensorial representation (Jimbo-type) of this algebra, which is used to classify the irreducible representations of \mathcal{E}_n , and plays a essential role in the proof of the linear independency of the basis proposed there.

Later, in [2] is proved that the algebra \mathcal{E}_n supports a Markov trace, this is achieved by using the method of relative traces, and the basis provided by Ryom-Hansen; for more examples of the relative traces method see [6], [8], [11]. Then, by using this trace as ingredient in the Jones's recipe [12], they define an invariant $\overline{\Delta}(u, A, B)$ for classical knots (respectively $\overline{\Gamma}(u, A, B)$ for singular knots) with parameters u, A and B . It's worth to say that for links, the invariant $\overline{\Delta}$ is more powerful than the HOMFLYPT polynomial, see [2, Addendum].

The bt-algebra have been studied by several researchers lately, which has helped to respond some important questions of its structure. In [7] J. Espinoza and Ryom-Hansen found a cellular basis for the bt-algebra, which is used to obtain an isomorphism theorem between \mathcal{E}_n and a sum of matrix algebras over certain wreath products. In her Ph.D. thesis [4] E. Banjo got an explicit isomorphism between the specialization $\mathcal{E}_n(1)$ and the small ramified partition algebra [16], and using it, she determined the complex generic representation of \mathcal{E}_n . Finally, in [15] I. Marin introduced a generalization of the bt-algebra, more precisely, given any Coxeter system (W, S) , he defined an extension of the corresponding Iwahori–Hecke algebra, denoted by C_W , which coincide with the algebra \mathcal{E}_n when W is the Coxeter group of type A.

Recently, in [8] we introduce a framization of the Hecke algebra of type B, denoted by $Y_{d,n}^B$, which is some kind of analogous of the Yokonuma–Hecke algebra for the B-type case, hence its notation. As we recalled above, the definition of the algebra $\mathcal{E}_n(u)$ is strongly related with certain subalgebra of $Y_{d,n}$. Then, it is natural try to define an analogue of the bt-algebra, this time, as a subalgebra of $Y_{d,n}^B$. Thus, in this paper we introduce a new algebra, denoted by $\mathcal{E}_n^B = \mathcal{E}_n^B(u)$, that contains the algebra of braids and ties, and we can say that it is a bt-algebra of type

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B. Moreover, we also construct a basis and a tensorial representation for it (adjusting the ideas given by Ryom-Hansen in [18]), having as goal to prove that \mathcal{E}_n^B supports a Markov trace, which is the main result of this work. It is important to note that the algebra introduced here doesn't coincide with the given by I. Marin considering W as the Coxeter group of B type, see Remark 1.

The article is organized as follows. In Section 2 we fix some notations and recall the results used in the paper. In Section 3, making an analogy with the classical case, we introduce the algebra \mathcal{E}_n^B , which contains the bt-algebra. Also, we give some relations that hold on it (which are, mostly, direct consequence of results from [8] and [18]), and we propose a diagrammatical interpretation for \mathcal{E}_n , in the sense of [2, Section 6]. In Section 4 we construct two linear bases for \mathcal{E}_n^B readjusting the ideas given in [18]. Similarly as in [8], one of these bases has a technical role, and the other is used for define a Markov trace in the last section. Moreover, we also construct a faithful tensorial representation of \mathcal{E}_n^B , which is the natural extension of the representation of the bt-algebra given by Ryom-Hansen, and plays a key role in the proof of the linear independency of one of the bases given here, see Theorem 2 (cf. [18, Theorem 3]). In Section 5 we prove that \mathcal{E}_n^B supports a Markov trace (Theorem 3), we prove that, constructing a family of relative traces using the basis given in the previous section. We use this method since as in the classical case, the basis obtained here cannot be defined in an inductive manner, then it is extremely difficult to define a Markov trace analogously to the Ocneanu's trace [12]. Thus, keeping the approach in [2], we split the proof of the main result in several lemmas, which give step by step the necessary conditions for the trace. Finally, using our trace as ingredient in Jones's recipe we define an invariant of classical links in the solid torus, which restricted to classical links (that is, braids of B-type without the 'loop generator' involved, see Section 2) coincide with $\bar{\Delta}$, and therefore it is more powerful than the Homflypt polynomial, whenever is evaluated in classical links.

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2. PRELIMINARIES

In this section we review known results, necessary for the sequel, and we also fix the following terminology and notations that will be used along the article:

- The letters u, v denote indeterminates. Consider $\mathbb{K} := \mathbb{C}(u, v)$.
- The term *algebra* means unital associative algebra over \mathbb{K} .
- The sets $\{0, 1, \dots, n\}$ and $\{1, \dots, n\}$ will be denoted simply by \mathbf{n}_0 and \mathbf{n} respectively.
- As usual, we denote by ℓ the length function associated to the Coxeter groups.

2.1. Set $n \geq 1$. Let us denote by W_n the Coxeter group of type B_n . This is the finite Coxeter group associated to the following Dynkin diagram

$$\begin{array}{ccccccc} \mathbf{r}_1 & & \mathbf{s}_1 & & & & \mathbf{s}_{n-2} & \mathbf{s}_{n-1} \\ \circ & \text{---} & \circ & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \circ \end{array}$$

Define $\mathbf{r}_k = \mathbf{s}_{k-1} \dots \mathbf{s}_1 \mathbf{r}_1 \mathbf{s}_1 \dots \mathbf{s}_{k-1}$ for $2 \leq k \leq n$. It is known, see [9], that every element $w \in W_n$ can be written uniquely as $w = w_1 \dots w_n$ with $w_k \in \mathbf{N}_k$, $1 \leq k \leq n$, where

$$\mathbf{N}_k := \{1, \mathbf{r}_k, \mathbf{s}_{k-1} \dots \mathbf{s}_i, \mathbf{s}_{k-1} \dots \mathbf{s}_i \mathbf{r}_i; 1 \leq i \leq k-1\}. \quad (1)$$

Moreover, this expression for w is reduced. Hence, we have $\ell(w) = \ell(w_1) + \dots + \ell(w_n)$.

Further, the group W_n can be realized as a subgroup of the permutation group of the set $X_n := \{-n, \dots, -2, -1, 1, 2, \dots, n\}$. Specifically, the elements of W_n are the permutations w such that $w(-m) = -w(m)$, for all $m \in X_n$. Then, the elements of W_n can be parameterized by the elements of $X_n^n := \{(m_1, \dots, m_n) \mid m_i \in X_n \text{ for all } i\}$ (see [10,

Lemma 1.2.1]). More precisely, the element $w \in W_n$ corresponds to the element $(m_1, \dots, m_n) \in X_n^n$ such that $m_i = w(i)$, for details see [8, Section 1.3].

The corresponding *braid group of type B_n* associated to W_n , is defined as the group \widetilde{W}_n generated by $\rho_1, \sigma_1, \dots, \sigma_{n-1}$ subject to the following relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| > 1, \\ \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1, \\ \rho_1 \sigma_i &= \sigma_i \rho_1 & \text{for } i > 1, \\ \rho_1 \sigma_1 \rho_1 \sigma_1 &= \sigma_1 \rho_1 \sigma_1 \rho_1. \end{aligned} \tag{2}$$

Geometrically, braids of type B_n can be viewed as classical braids of type A_n with $n + 1$ strands, such that the first strand is identically fixed. This is called ‘the fixed strand’. The 2nd, \dots , $(n + 1)$ st strands are renamed from 1 to n and they are called ‘the moving strands’. The ‘loop’ generator ρ_1 stands for the looping of the first moving strand around the fixed strand in the right-handed sense, see [13, 14]. In Figure 1 we illustrate a braid of type B₄.



FIGURE 1. A braid of B₄-type.

2.2. Recently in [8] we introduced a new framization of the Hecke algebra of type B, denoted by $Y_{d,n}^B := Y_{d,n}^B(u, v)$. This algebra was constructed searching an analogous of the Yokonuma–Hecke algebra for the type B case, this, with the final objective to explore their usefulness in knot theory. Thus, in this article we constructed two linear bases, a faithful tensorial representation of Jimbo type for $Y_{d,n}^B(u, v)$, and we proved that $Y_{d,n}^B$ supports a Markov trace. Finally we defined, by using Jones’s recipe, a new invariant for framed and classical links in the solid torus. Along this paper we use several properties of this algebra, then we recall some of them. We begin with its definition

Definition 1. Let $n \geq 2$. The algebra $Y_{d,n}^B := Y_{d,n}^B(u, v)$, is defined as the algebra over $\mathbb{K} := \mathbb{C}(u, v)$ generated by framing generators t_1, \dots, t_n , braiding generators g_1, \dots, g_{n-1} and the loop generator b_1 , subject to the following relations

$$g_i g_j = g_j g_i \quad \text{for } |i - j| > 1, \quad (3)$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{for } |i - j| = 1, \quad (4)$$

$$b_1 g_i = g_i b_1 \quad \text{for all } i \neq 1, \quad (5)$$

$$b_1 g_1 b_1 g_1 = g_1 b_1 g_1 b_1, \quad (6)$$

$$t_i t_j = t_j t_i \quad \text{for all } i, j, \quad (7)$$

$$t_j g_i = g_i t_{s_i(j)} \quad \text{for all } i, j, \quad (8)$$

$$t_i b_1 = b_1 t_i \quad \text{for all } i, \quad (9)$$

$$t_i^d = 1 \quad \text{for all } i, \quad (10)$$

$$g_i^2 = 1 + (u - u^{-1})e_i g_i \quad \text{for all } i, \quad (11)$$

$$b_1^2 = 1 + (v - v^{-1})f_1 b_1, \quad (12)$$

where

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s} \quad \text{and} \quad f_j := \frac{1}{d} \sum_{s=0}^{d-1} t_j^s; \quad \text{for } 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq n.$$

For $n=1$, we define the algebra $Y_{d,1}^B$ as the algebra generated by $1, b_1$ and t_1 satisfying the relations (9), (10) and (12).

Notice that the elements f_j 's and e_i 's are idempotents. Also, It is clear that the element f_1 commutes with b_1 and e_i commutes with g_i . These facts imply that the generators b_1 and the g_i 's are invertible. Namely, we have:

$$b_1^{-1} = b_1 - (v - v^{-1})f_1 \quad \text{and} \quad g_i^{-1} = g_i - (u - u^{-1})e_i. \quad (13)$$

Now we recall the two basis of $Y_{d,n}^B$ given in [8], which we will use for the sequel.

Set $\bar{b}_1 := b_1$, $\bar{b}_k := g_{k-1} \dots g_1 b_1 g_1 \dots g_{k-1}$, and $b_k := g_{k-1} \dots g_1 b_1 g_1^{-1} \dots g_{k-1}^{-1}$ for all $2 \leq k \leq n$. For all $1 \leq k \leq n$, let us define inductively the sets $N_{d,k}$ by $N_{d,1} := \{t_1^m, \bar{b}_1 t_1^m; 0 \leq m \leq d-1\}$ and

$$N_{d,k} := \{t_k^m, \bar{b}_k t_k^m, g_{k-1} x; x \in N_{d,k-1}, 0 \leq m \leq d-1\} \quad \text{for all } 2 \leq k \leq n.$$

Analogously, for all $1 \leq k \leq n$ we define inductively the sets $M_{d,k}$ exactly like $N_{d,k}$'s but exchanging \bar{b}_k by b_k in every case.

Finally, consider $D_n = \{n_1 n_2 \dots n_n \mid n_i \in N_{d,i}\}$ and $C_n = \{m_1 m_2 \dots m_n \mid m_i \in M_{d,i}\}$. Then, we have that D_n and C_n are bases of $Y_{d,n}^B$, for details see [8, Section 4].

2.3. We denote by $P(\mathbf{n})$ the set formed by the set-partitions of \mathbf{n} , recall that the cardinality of $P(\mathbf{n})$ is the n -th Bell number, denoted by b_n . The subsets of \mathbf{n} entering a partition are called blocks. For short we shall omit the subset of cardinality 1 (single blocks) in the partition. For example, the partition $I = (\{1, 2, 3\}, \{4, 6\}, \{5\}, \{7\})$ in $P(\mathbf{7})$, will be simply written as $I = (\{1, 2, 3\}, \{4, 6\})$. Moreover, $\text{Supp}(I)$ will denote the union of non-single blocks of I .

The symmetric group S_n acts naturally on $P(\mathbf{n})$. More precisely, set $I = (I_1, \dots, I_m) \in P(\mathbf{n})$. The action $w(I)$ for a $w \in S_n$ is given by

$$w(I) = ((w(I_1), \dots, w(I_m)))$$

where $w(I_k)$ is the set obtained of applying w to the set I_k .

The pair $(P(\mathbf{n}), \preceq)$ is a poset. Specifically, given $I = (I_1, \dots, I_m)$, $J = (J_1, \dots, J_s) \in P(\mathbf{n})$, the partial order \preceq is defined by.

$$I \preceq J \quad \text{if and only if } J \text{ is a union of some blocks of } I$$

When $I \preceq J$, we will say that I refines J .

Let $I, J \in \mathcal{P}(\mathbf{n})$, we denote $I * J$ the minimal set partition refined by I and J . Let A a subset of \mathbf{n} . Along the work we will use for short $I * A$ instead $I * (A)$. Thus, if $I = (I_1, \dots, I_k, I_{i_{k+1}}, \dots, I_{i_m})$, where the first k blocks are the blocks that have intersection with A , and the rest are those that don't have. Then $I * A$ is given by

$$I * A = (A', I_{k+1}, \dots, I_m)$$

where $A' = A \cup I_1 \cup \dots \cup I_k$. In particular, $I * \{j, m\}$ coincides with I if j and m already belong to the same block, otherwise, $I * \{j, m\}$ coincides with I except for the blocks containing j and m , which merge in a sole block. For short, we will write $I * j$ instead of $I * \{j, j+1\}$. For instace, for the set partition $I = (\{1, 4\}, \{2, 5\}, \{3, 6, 7\})$ in $\mathcal{P}(8)$:

$$I * \{4, 5, 8\} = (\{1, 2, 4, 5, 8\}, \{3, 6, 7\}) \quad , \quad I * 2 = (\{1, 4\}, \{2, 3, 5, 6, 7\}) \quad \text{and} \quad I * 6 = I$$

Also, for $I \in \mathcal{P}(\mathbf{n})$, we denote $I \setminus n$ the element in $\mathcal{P}(\mathbf{n} - 1)$ that is obtained by removing n from I . Let be $\mathcal{P}(\mathbf{n}_0)$ the set of partitions of \mathbf{n}_0 , note that, $\mathcal{P}(\mathbf{n}_0)$ is essentially $\mathcal{P}(\mathbf{n} + 1)$, then all the definitions and notations above are valid for partitions in $\mathcal{P}(\mathbf{n}_0)$. Finally, for $A \subseteq \mathbf{n}_0$ we define $A^* = A \setminus \{0\}$.

3. AN ALGEBRA OF BRAIDS AND TIES INSIDE $\mathcal{Y}_{d,n}^B$

In this section we propose a generalization of the algebra of braids and ties $\mathcal{E}_n(u)$ defined originally in [1] and posteriorly studied in [2, 3]. As we note previously, the definition of $\mathcal{E}_n(u)$ definition was obtained by considering abstractly as a subalgebra of $\mathcal{Y}_{d,n}(u)$. In [8] we introduce a framization of the Hecke algebra of type B, denoted by $\mathcal{Y}_{d,n}^B$, which is the some kind of analogous of the Yokonuma–Hecke algebra for the B-type case. Then, it is natural to define an analogue of the bt–algebra, this time, considering a subalgebra of $\mathcal{Y}_{d,n}^B$, which carries us to the next definition.

Definition 2. Let $n \geq 2$. We define the bt–algebra of type B, denoted by $\mathcal{E}_n^B = \mathcal{E}_n^B(u, v)$, as the algebra generated by B_1, T_1, \dots, T_{n-1} and $F_1, \dots, F_n, E_1, \dots, E_{n-1}$, subject to the following relation

$$T_i T_j = T_j T_i \quad \text{for all } |i - j| > 1 \quad (14)$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for all } 1 \leq i \leq n - 2 \quad (15)$$

$$T_i^2 = 1 + (u - u^{-1}) E_i T_i \quad \text{for all } 1 \leq i \leq n - 1 \quad (16)$$

$$E_i^2 = E_i \quad \text{for all } i \quad (17)$$

$$E_i E_j = E_j E_i \quad \text{for all } i, j \quad (18)$$

$$E_i T_i = T_i E_i \quad \text{for all } 1 \leq i \leq n - 1 \quad (19)$$

$$E_i T_j = T_j E_i \quad \text{for all } |i - j| > 1 \quad (20)$$

$$E_i E_j T_i = T_i E_i E_j = E_j T_i E_j \quad \text{for all } 1 \leq i, j \leq n - 1 \text{ such that } |i - j| = 1 \quad (21)$$

$$E_i T_j T_i = T_j T_i E_j \quad \text{for all } 1 \leq i, j \leq n - 1 \text{ such that } |i - j| = 1 \quad (22)$$

$$B_1 T_1 B_1 T_1 = T_1 B_1 T_1 B_1 \quad (23)$$

$$B_1 T_i = T_i B_1 \quad \text{for all } i > 1 \quad (24)$$

$$B_1^2 = 1 + (v - v^{-1}) F_1 B_1 \quad (25)$$

$$B_1 E_i = E_i B_1 \quad \text{for all } i \quad (26)$$

$$F_i^2 = F_i \quad \text{for all } i \quad (27)$$

$$B_1 F_j = F_j B_1 \quad \text{for all } j \quad (28)$$

$$F_i E_j = E_j F_i \quad \text{for all } i, j \quad (29)$$

$$F_j T_i = T_i F_{s_i(j)} \quad \text{where } s_i \text{ is the transposition } (i, i + 1) \quad (30)$$

$$E_i F_i = F_i F_{i+1} = E_i F_{i+1} \quad \text{for all } 1 \leq i \leq n - 1 \quad (31)$$

For $n = 1$ we define the algebra \mathcal{E}_1^B as the algebra generated by $1, B_1$ and F_1 subject to the relations (25),(27) and (28).

More precisely, the definition of \mathcal{E}_n^B is obtained by considering abstractly the subalgebra of $Y_{d,n}^B$ generated by b_1 , and the elements g_i 's, e_i 's and f_i 's. Thus, considering g_i as T_i , e_i as E_i , f_i as F_i and b_1 as B_1 , the defining relations of \mathcal{E}_n^B correspond to the set of relations derived from the relations (3)-(12) of $Y_{d,n}^B$.

Proposition 1. *There is a natural homomorphism $\varphi_n : \mathcal{E}_n^B \rightarrow Y_{d,n}^B$ defined by the mapping $T_i \mapsto g_i$, $E_i \mapsto e_i$, $F_i \mapsto f_i$ and $B_1 \mapsto b_1$.*

Remark 1. Let be (W, S) a Coxeter system, and C_W the algebra introduced by I. Marin in [15]. It is known that $C_W = \mathcal{E}_n$ when W is the Coxeter group of type A_{n-1} , then, it is natural to think that C_W should coincide with \mathcal{E}_n^B when $W = W_n$, but this doesn't happen. In fact, we will prove in Section 4 that the dimension of \mathcal{E}_n^B is $b_{n+1}|W_n|$, meanwhile C_W has dimension $Bell(W)|W|$, where the $Bell(W)$ is a entire number called the Bell number of W (by obvious reasons). These numbers are not completely determined for type B_n , but are known for low dimensions, more precisely, for $n \geq 2$ the sequence of dimensions is the following: 8, 38, 218, 1430, 10514, ...; for details see [15, Section 3.6]. Thus, we have that these algebras are different, which indicates that the algebra \mathcal{E}_n^B should be interesting by itself.

Remark 2. Note that the quadratic relation for $Y_{d,n}$ and \mathcal{E}_n used in [1, 2] is different than the used here. However, it is well known that modifying the set of generators of $Y_{d,n}$ (respectively \mathcal{E}_n), it is possible to get a presentation with the desired quadratic relation. More precisely, if \tilde{g}_i (respectively \tilde{T}_i) denote the original braid generators of the Yokonuma–Hecke algebra (respectively, of the bt-algebra), and u the parameter used in the usual quadratic relation. Then, by taking $u = u^2$ and $g_i = \tilde{g}_i + (u^{-1} - 1)e_i\tilde{g}_i$ (respectively $T_i = \tilde{T}_i + (u^{-1} - 1)e_i\tilde{T}_i$), we obtain a presentation including the quadratic relation used by us, cf. [5, Remark 1]. Consequently, the bt-algebra \mathcal{E}_n can be regarded as the algebra generated by the elements T_i 's and E_i 's subject to the relations (14)-(22), thus, in particular, we have that $\mathcal{E}_n \subseteq \mathcal{E}_n^B$.

Proposition 2. *The map $\varphi : \mathcal{E}_n^B \rightarrow \mathcal{E}_n$ given by $\varphi(T_i) = T_i$, $\varphi(E_i) = E_i$, $\varphi(B_1) = 1$ and $\varphi(F_i) = 1$ define a natural epimorphism.*

Now, we recall some useful result from [18]. For $i < j$, we define $E_{i,j}$ by

$$E_{i,j} = \begin{cases} E_i & \text{for } j = i + 1 \\ T_i \dots T_{j-2} E_{j-1} T_{j-2}^{-1} \dots T_i^{-1} & \text{otherwise} \end{cases}$$

For a nonempty subset J of \mathbf{n} we define $E_J = 1$ if $|J| = 1$ and

$$E_J := \prod_{(i,j) \in J \times J, i < j} E_{i,j}$$

Note that $E_{\{i,j\}} = E_{i,j}$. Also we have by [18, Lemma 4] that

$$E_j = \prod_{j \in J, j \neq i_0} E_{i_0,j}, \quad \text{where } i_0 = \min(J)$$

Moreover, for $I = (I_1, \dots, I_m) \in P(\mathbf{n})$ we define E_I by

$$E_I = \prod_k E_{I_k}.$$

In the same fashion, for any subset A of \mathbf{n} we define

$$F_A = \prod_{i \in A} F_i$$

Remark 3. Notice that by (30) we have that

$$F_i = T_{i-1} \dots T_1 F_1 T_1^{-1} \dots T_{i-1}^{-1} = T_{i-1}^{-1} \dots T_1^{-1} F_1 T_1 \dots T_{i-1}. \quad (32)$$

Then, we can omit F_2, \dots, F_n from the presentation (changing/adding certain relations simultaneously!), but we prefer include them, since the computations become simpler by using them. Also note that, using (30), we can obtain a generalization of (31) by conjugating it. More precisely we have

$$E_{i,j} F_i = E_{i,j} F_j = F_i F_j \quad \text{for all } 1 \leq i < j \leq n \quad (33)$$

Now, we introduce certain elements that will be used for the construction of a linear basis of \mathcal{E}_n^B . Let be $\overline{B}_1 = B_1$, and for $k \geq 2$ we define

$$\overline{B}_k := T_{k-1} \dots T_1 B_1 T_1 \dots T_{k-1}$$

$$B_k := T_{k-1} \dots T_1 B_1 T_1^{-1} \dots T_{k-1}^{-1}$$

Further, we consider the sets M_k defined inductively by $M_1 := \{1, B_1\}$ and

$$M_k := \{1, B_k, T_{k-1}x \mid x \in M_{k-1}\} \quad \text{for all } 2 \leq k \leq n.$$

Analogously, for all $1 \leq k \leq n$ we define inductively the sets N_k 's exactly like M_k 's but exchanging B_k by \overline{B}_k in each case.

Now notice that every element of M_k has the form $\mathbb{T}_{k,j}^+$ or $\mathbb{T}_{k,j}^-$ with $j \leq k$, where

$$\mathbb{T}_{k,k}^+ := 1, \quad \mathbb{T}_{k,j}^+ := T_{k-1} \dots T_j \quad \text{for } j < k,$$

and

$$\mathbb{T}_{k,k}^- := B_k, \quad \mathbb{T}_{k,j}^- := T_{k-1} \dots T_j B_j \quad \text{for } j < k.$$

Similar expressions exist for elements in N_k exchanging B_k by \overline{B}_k as well, which will be denoted by $\overline{\mathbb{T}}_{k,j}^+$ and $\overline{\mathbb{T}}_{k,j}^-$.

The following results are direct consequence of [8], and these will be used frequently in the sequel

Lemma 1. For $n \geq 2$ the following relations hold

$$\text{i) } \overline{\mathbb{T}}_{n,k}^- T_j = \begin{cases} T_j \overline{\mathbb{T}}_{n,k}^- & j < k-1 \\ \overline{\mathbb{T}}_{n,k-1}^- + (u - u^{-1}) \overline{\mathbb{T}}_{n,k}^- E_j & j = k-1 \\ \overline{\mathbb{T}}_{n,k+1}^- & j = k \\ T_{j-1} \overline{\mathbb{T}}_{n,k}^- & j > k \end{cases}$$

$$\text{ii) } \mathbb{T}_{n,k}^+ T_j = \begin{cases} T_j \mathbb{T}_{n,k}^+ & j < k-1 \\ \mathbb{T}_{n,k-1}^+ & j = k-1 \\ \mathbb{T}_{n,k+1}^+ + (u - u^{-1}) \mathbb{T}_{n,k}^+ E_j & j = k \\ T_{j-1} \mathbb{T}_{n,k}^+ & j > k \end{cases}$$

$$\text{iii) } \overline{\mathbb{T}}_{n,k}^- B_1 = \begin{cases} \overline{\mathbb{T}}_{n,k}^+ + (v - v^{-1}) \overline{\mathbb{T}}_{n,k}^- F_1 & \text{for } k = 1 \\ B_1 \overline{\mathbb{T}}_{n,k}^- & \text{for } k \neq 1 \end{cases}$$

$$\text{iv) } \mathbb{T}_{n,k}^+ B_1 = \begin{cases} \mathbb{T}_{n,k}^- & \text{for } k = 1 \\ B_1 \mathbb{T}_{n,k}^+ & \text{for } k \neq 1 \end{cases}$$

In particular, we have that

$$\begin{aligned}
- \bar{B}_n T_j &= \begin{cases} T_j \bar{B}_n & \text{for } j < n-1 \\ T_{n-1} \bar{B}_{n-1} + (u - u^{-1}) \bar{B}_n E_{n-1} & \text{for } j = n-1 \end{cases} \\
- \bar{B}_n T_{n-1} &= T_{n-1} \bar{B}_{n-1} + (u - u^{-1}) \bar{B}_n E_{n-1}
\end{aligned}$$

Proof. See [8, Lemma 3 and 4]. \square

Lemma 2. For $n \geq 2$ the following relations hold

$$\begin{aligned}
\text{i) } \mathbb{T}_{n,k}^\pm T_j &= \begin{cases} T_j \mathbb{T}_{n,k}^\pm & j < k-1 \\ \mathbb{T}_{n,k-1}^\pm & j = k-1 \\ \mathbb{T}_{n,k+1}^\pm + (u - u^{-1}) \mathbb{T}_{n,k}^\pm E_j & j = k \\ T_{j-1} \mathbb{T}_{n,k}^\pm & j > k \end{cases} \\
\text{ii) } \mathbb{T}_{n,k}^- B_1 &= \begin{cases} \mathbb{T}_{n,k}^+ + (v - v^{-1}) \mathbb{T}_{n,k}^- F_1 & \text{for } k = 1 \\ B_1 \mathbb{T}_{n,k}^- + (u - u^{-1}) \alpha_{n,k} & \text{for } k \neq 1 \end{cases} \\
\text{iii) } \mathbb{T}_{n,k}^+ B_1 &= \begin{cases} \mathbb{T}_{n,k}^- & \text{for } k = 1 \\ B_1 \mathbb{T}_{n,k}^+ & \text{for } k \neq 1 \end{cases}
\end{aligned}$$

where $\alpha_{n,k} = (B_1 T_1^{-1} \dots T_{k-2}^{-1} \mathbb{T}_{n,1}^- E_{1,k} - T_1^{-1} \dots T_{k-2}^{-1} \mathbb{T}_{n,1}^- B_1 E_{1,k})$. In particular, we have

$$\begin{aligned}
- B_n T_{n-1} &= T_{n-1} B_{n-1} \\
- B_n T_j &= T_j B_n, \text{ for all } j < n-1 \\
- B_n B_1 &= B_1 B_n + (u - u^{-1}) [B_1 T_1^{-1} \dots T_{n-2}^{-1} \mathbb{T}_{n,1}^- E_{1,k} - T_1^{-1} \dots T_{n-2}^{-1} \mathbb{T}_{n,1}^- B_1 E_{1,k}]
\end{aligned}$$

Proof. See [8, Lemma 5,6 and 7]. \square

Lemma 3. The following claims hold in \mathcal{E}_n^B .

- (i) $T_k B_k B_{k+1} = B_k T_k B_k$, for $k \geq 1$.
- (ii) $\mathbb{T}_{k,j}^- B_k = B_{k-1} \mathbb{T}_{k,j}^-$, for $k \geq 2$.

Proof. See (iv) [8, Proposition 2] and (i) [8, Lemma 8] respectively. \square

The defining generators B_1 and T_i 's of the algebra \mathcal{E}_n^B satisfy the same braid relations as the Coxeter generators \mathbf{r}_1 and \mathbf{s}_i of the group W_n . Thus, the well-known Matsumoto's Lemma implies that if $w_1 \dots w_m$ is a reduced expression of $w \in W_n$, with $w_i \in \{\mathbf{r}_1, \mathbf{s}_1, \dots, \mathbf{s}_{n-1}\}$, then the following element T_w is well-defined:

$$T_w := T_{w_1} \dots T_{w_m}, \quad (34)$$

where $T_{w_i} = B_1$, if $w_i = \mathbf{r}_1$ and $T_{w_i} = T_j$, if $w_i = \mathbf{s}_j$. Therefore, according to 2.1 we have that $\{T_w \mid w \in W_n\} = \{\mathbf{r}_1 \dots \mathbf{r}_n \mid \mathbf{r}_i \in \mathbf{N}_i\}$. In the same fashion, we have a natural bijection $\gamma : \{\mathbf{m}_1 \dots \mathbf{m}_n \mid \mathbf{m}_i \in \mathbf{M}_i\} \rightarrow W_n$, induced by the map $B_k \mapsto \mathbf{r}_k$, $T_i \mapsto \mathbf{s}_i$.

Let $w \in W_n$, and $\eta : W_n \rightarrow S_n$ the natural projection defined by $\mathbf{r}_1 \mapsto 1$ and $\mathbf{s}_i \mapsto s_i$. Since in \mathcal{E}_n^B the action of B_1 over the elements F_i 's and E_i 's is trivial, that is, these commute, we have the next result

Lemma 4. Let $w \in W_n$, $I \in \mathbf{P}(\mathbf{n})$ and $A \subseteq \mathbf{n}$, then

- a) $T_w E_I T_w^{-1} = E_{\bar{w}(I)}$
- b) $T_w F_A T_w^{-1} = F_{\bar{w}(A)}$

where $\bar{w} := \eta(w)$

Proof. For a) see [18, Corollary 1], and for b) the result follows by applying the defining relations (30) and (28). \square

Corollary 1. *Let $v \in \{m_1 \dots m_n \mid m_i \in M_i\}$, $I \in P(\mathbf{n})$ and $A \subseteq \mathbf{n}$, then*

- a) $vE_I v_{-1} = E_w(I)$
- b) $vF_A v_{-1} = F_w(A)$

where $w = \eta \circ \gamma(v)$.

3.1. Diagrams for \mathcal{E}_n^B . We associate to each word in the algebra \mathcal{E}_n^B , a tied braid of type B_n according the following identifications. For $n \geq 1$ we associate the unit of \mathcal{E}_n^B with the trivial braid of type B_n , B_1 with the “loop” generator, and F_j with the braid of type B_n whose has a tied between the fixed strand and the j -th moving strand. For $n \geq 2$, as in the classical case we associate T_i with the usual braid generator, and E_i with the B-type braid whose has a tied between the i -th and $i+1$ -st moving strand. We can see this identification in the following figure

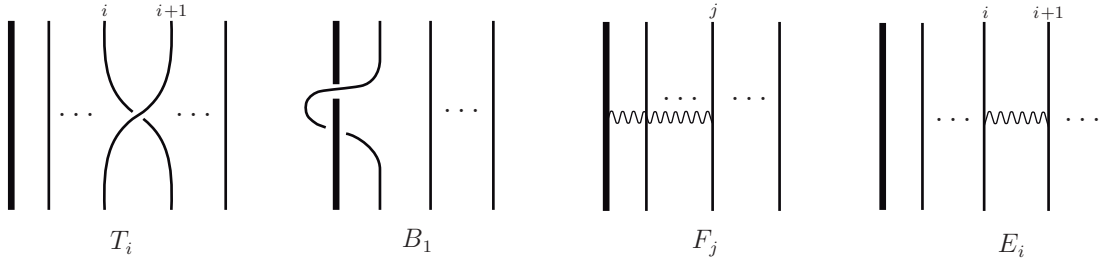


FIGURE 2. Diagrammatic interpretation of the generators of \mathcal{E}_n^B

We consider the multiplication of diagrams by concatenation, that is, given two diagrams d_1 and d_2 , the multiplication $d_1 d_2$ is the diagram that result from putting the diagram d_2 below to the diagram d_1 , obtaining pictures like in Figure 3. Thus, we have a diagrammatic interpretation for every word in \mathcal{E}_n^B .

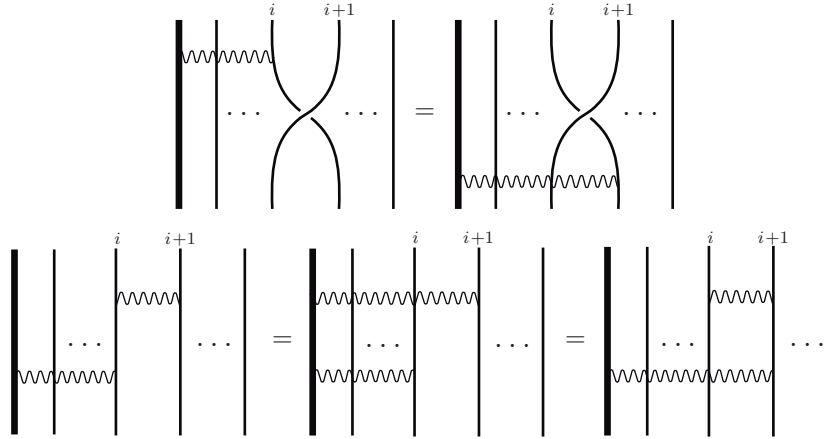


FIGURE 3. Relations (30) and (31) in terms of diagrams

Remark 4. The previous identification provide an epimorphism δ , from the algebra \mathcal{E}_n^B to an algebra of diagrams, which will be denoted by $\widetilde{\mathcal{E}_n^B}$. This algebra is generated by the elements in Figure 2, and satisfies the defining relations of \mathcal{E}_n^B rewritten as diagram relations, for instance see Figure 3. As we would expected, the ties of the elements of $\widetilde{\mathcal{E}_n^B}$ keep all the properties explained in [2, Section 6], that is, the *elasticity*, *transparency* and *transitivity*.

For example, the elasticity and transparency properties, for ties involving the moving strands are inherited by the relations in common with the bt-algebra, and for the ties attached to the fixed strand are guaranteed by relation (30), see Figure 4. On the other hand, the transitivity property is a consequence of Proposition 3 proven in the next section.

In the same fashion, we can conjecture that the homomorphism δ is, in fact, an isomorphism. However, to give a formal proof of this fact, is in general a problem itself, for example see [17]. Therefore we will continue without proving it, since we would deviate of our main goal.

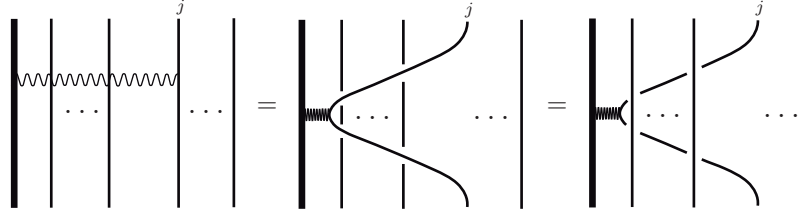


FIGURE 4. Equation (32) expressed in diagrams

4. A LINEAR BASIS FOR \mathcal{E}_n^B

In this section we introduce two linear bases for the algebra \mathcal{E}_n^B . One of them will be used to define a Markov trace in the next section, and the other for proving the linear independence of the first one. Additionally we give a faithful tensorial representation of the algebra \mathcal{E}_n^B based in the constructed for $Y_{d,n}^B$ in [8, Section 3].

We begin the section proving a technical result, which will be useful to set properly our base.

Let be $I, I' \in P(\mathbf{n})$ and $A, A' \in \mathbf{n}$. Note that, the elements $E_I F_A, E_{I'} F_{A'} \in \mathcal{E}_n^B$ can be equal even when $I \neq I'$ and $A \neq A'$. For instance, let $I = (\{2, 3, 5\}, \{4, 6\})$, $I' = (\{4, 6\}) \in P(\mathbf{n})$ and $A = \{3\}$, $A' = \{2, 3, 5\}$. Then, we have

$$\begin{aligned} E_I F_A &= E_{2,3} E_{2,5} E_{4,6} F_3 \\ &= F_3 E_{2,3} E_{2,5} E_{4,6} \\ &= F_3 F_2 E_{2,5} E_{4,6} \\ &= F_2 F_3 F_5 E_{4,6} \\ &= E_{I'} F_{A'} \end{aligned}$$

by (33). Using the same idea we can prove the following result

Proposition 3. *The set $\mathcal{A}_n = \{E_I F_A \mid I \in P(\mathbf{n}), A \subseteq \mathbf{n}\}$ is parameterized by $P(\mathbf{n}_0)$. That is, there is a bijection between the sets $P(\mathbf{n}_0)$ and \mathcal{A}_n .*

Proof. Let be $I = (I_1, \dots, I_m)$ a partition in $P(\mathbf{n}_0)$. We define $\Psi : P(\mathbf{n}_0) \rightarrow \mathcal{A}_n$ as follows

$$\Psi(I) = \begin{cases} E_I & \text{if } 0 \notin I_j \text{ for all } 1 \leq j \leq m \\ E_{I \setminus I_k} F_{I_k^*} & \text{if } 0 \in I_k \text{ for some } 1 \leq k \leq m \end{cases}$$

where $I \setminus I_k$ denote the partition obtained by removing the block I_k from I . For the other hand, let be $I \in P(\mathbf{n})$, $A \in \mathbf{n}$ we define $\varphi : \mathcal{A}_n \rightarrow P(\mathbf{n}_0)$ as follows

$$\varphi(E_I F_A) = I * A^0$$

where $A^0 = A \cup \{0\}$ and I is considered as a element of $P(\mathbf{n}_0)$ (since $P(\mathbf{n}) \subseteq P(\mathbf{n}_0)$). It is not difficult to prove that $\varphi \circ \Psi = Id_{P(\mathbf{n}_0)}$ and $\Psi \circ \varphi = Id_{\mathcal{A}_n}$. In fact, let say that $I = (I_1, \dots, I_k)$, without lose generality consider

that I_1, \dots, I_r are the blocks of I that have intersection with A , and I_{r+1}, \dots, I_k the blocks that don't have. By definition, we have that

$$I * A^0 = (C, I_{r+1}, \dots, I_k)$$

where $C = A^0 \cup I_1 \cup \dots \cup I_r$. Thus, $\Psi(I * A^0) = F_{C^*} E_{I \setminus A}$ where $I \setminus A$ is the partition (I_{r+1}, \dots, I_k) in $P(\mathbf{n})$.

Then, it is enough to prove that

$$F_{C^*} E_{I \setminus A} = E_I F_A. \quad (35)$$

First, recall that for any block I_r , we have that

$$E_{I_r} = \prod_{j \neq i_0, j \in I_r} E_{i_0, j}, \quad \text{where } i_0 = \min I_r$$

Thus, (33) implies that

$$E_{I_r} F_j = E_{I_r} F_{i_0}$$

for any $j \in A \cap I_r$. Therefore, more generally using (27) we have

$$\begin{aligned} E_{I_r} F_A &= E_{I_r} F_{i_0} F_{A \setminus I_r} \\ &= \prod_{j \neq i_0} E_{i_0, j} F_{i_0} F_{A \setminus I_r} \\ &= \prod_{j \neq i_0} F_{i_0} F_j F_{A \setminus I_r} \\ &= F_{I_r} F_{A \setminus I_r} = F_{A \cup I_r} \end{aligned}$$

Finally, (35) follows by using this argument for every block that has intersection with A . The converse can be proven from an analogous way. \square

From now on, given a partition $I \in P(\mathbf{n}_0)$ we will denote the element $\Psi(I)$ by EF_I .

Proposition 4. *The set $\mathcal{B}_n = \{EF_I T_w \mid I \in P(\mathbf{n}_0) \text{ and } w \in W_n\}$ spans the algebra \mathcal{E}_n^B .*

Proof. We proceed as in [8, Proposition 4], that is, we will prove by induction that \mathfrak{B}_n , the linear subspace of \mathcal{E}_n^B spanned by \mathcal{B}_n is equal to \mathcal{E}_n^B . The assertion is clear for $n = 1$. Assume now that \mathcal{E}_{n-1}^B is spanned by \mathfrak{B}_{n-1} . Notice that $1 \in \mathfrak{B}_n$. This fact and proving that \mathfrak{B}_n is a right ideal, implies the proposition. Now, we deduce that \mathfrak{B}_n is a right ideal from the hypothesis induction and Lemma 1. Indeed, let $EF_I T_w$ a element in \mathcal{B}_n and Y a generator of \mathcal{E}_n^B , and suppose that $n \in \text{Supp}(I)$, we have that $EF_I T_w Y$ is equal to

$$E_{n,i} EF_{I \setminus n} x \mathbb{T}_{n,k}^\pm Y \quad \text{or} \quad F_n EF_{I \setminus n} x \mathbb{T}_{n,k}^\pm Y, \quad \text{where } x \in \langle B_1, T_1, \dots, T_{n-2} \rangle, \text{ and } i = \min(I_r), \text{ with } n \in I_r.$$

depending if n belongs to I_0 or not. Then, using the relations from Lemma 1 we can convert this expressions in a linear combination of elements of the form

$$E_{n,i}(X) \mathbb{T}_{n,k'}^\pm \quad \text{or} \quad F_n(X) \mathbb{T}_{n,k'}^\pm, \quad \text{where } X \in \mathcal{E}_{n-1}^B.$$

thus, the result follows by applying the induction hypothesis. The case when $n \notin \text{Supp}(I)$ will be omitted, since it follows by analogous way. \square

Now, we will focus in proving the linear independence of the set \mathcal{B}_n , to achieve that, we will readjust the arguments used by Ryom-Hansen in [18]. For that, we use a tensorial representation of \mathcal{E}_n^B , which is obtained by restricting the representation of $Y_{d,n}^B$ constructed in [8], considering $d = n + 1$, to the subalgebra $\varphi_n(\mathcal{E}_n^B)$.

More precisely, let V be a \mathbb{K} -vector space with basis $\mathfrak{B} = \{v_i^r; i \in X_n, 0 \leq r \leq n\}$. As usual we denote by $\mathfrak{B}^{\otimes k}$ the natural basis of $V^{\otimes k}$ associated to \mathfrak{B} . That is, the elements of $\mathfrak{B}^{\otimes k}$ are of the form:

$$v_{i_1}^{m_1} \otimes \cdots \otimes v_{i_k}^{m_k}$$

where $(i_1, \dots, i_k) \in X_n^k$ and $(m_1, \dots, m_k) \in \mathbf{n}^k$.

We define the endomorphisms $\mathbf{F}, \mathbf{B} : V \rightarrow V$ by:

$$(v_i^r)\mathbf{B} = \begin{cases} v_{-i}^r & \text{for } i > 0 \text{ and } r = 0, \\ v_{-i}^r + (\mathbf{v} - \mathbf{v}^{-1})v_i^r & \text{for } i < 0 \text{ and } r = 0, \\ v_{-i}^r & \text{for } r \neq 0. \end{cases}$$

and

$$(v_i^r)\mathbf{F} = \begin{cases} 0 & r > 0 \\ v_i^r & r = 0 \end{cases}$$

On the other hand we define $\mathbf{T}, \mathbf{E} : V \otimes V \rightarrow V \otimes V$ by

$$(v_i^r \otimes v_j^s)\mathbf{T} = \begin{cases} \mathbf{u}v_j^s \otimes v_i^r & \text{for } i = j \text{ and } r = s, \\ v_j^s \otimes v_i^r & \text{for } i < j \text{ and } r = s, \\ v_j^s \otimes v_i^r + (\mathbf{u} - \mathbf{u}^{-1})v_i^r \otimes v_j^s & \text{for } i > j \text{ and } r = s, \\ v_j^s \otimes v_i^r & \text{for } r \neq s. \end{cases}$$

and

$$(v_i^r \otimes v_j^s)\mathbf{E} = \begin{cases} 0 & r \neq s \\ v_i^r \otimes v_j^s & r = s \end{cases}$$

For all $1 \leq i \leq n-1, 1 \leq j \leq n$ we extend these endomorphisms to the endomorphisms $\mathbf{E}_i, \mathbf{T}_i, \mathbf{B}_1, \mathbf{F}_j$ of the n -th tensor power $V^{\otimes n}$ of V , as follows:

$$\begin{aligned} \mathbf{E}_i &:= 1_V^{\otimes(i-1)} \otimes \mathbf{E} \otimes 1_V^{\otimes(n-i-1)}, & \mathbf{B}_1 &:= \mathbf{B} \otimes 1_V^{\otimes(n-1)}, \\ \mathbf{T}_i &:= 1_V^{\otimes(i-1)} \otimes \mathbf{T} \otimes 1_V^{\otimes(n-i-1)}, & \mathbf{F}_j &:= 1_V^{\otimes(j-1)} \otimes \mathbf{F} \otimes 1_V^{\otimes(n-j)} \end{aligned}$$

where $1_V^{\otimes k}$ denotes the endomorphism identity of $V^{\otimes k}$.

Theorem 1. *The mapping $B_1 \mapsto \mathbf{B}_1, T_i \mapsto \mathbf{T}_i, E_i \mapsto \mathbf{E}_i$ and $F_i \mapsto \mathbf{F}_i$ defines a representation Φ of $\mathcal{E}_n^{\mathbf{B}}$ in $\text{End}(V^{\otimes n})$.*

Proof. It is a consequence of [8, Theorem 1]. □

Further, we have

Proposition 5. (See [8, Proposition 3]) *Let $w \in W_n$ parameterized by $(m_1, \dots, m_n) \in X_n^n$. Then*

$$(v_1^{r_1} \otimes \cdots \otimes v_n^{r_n})\Phi_w = v_{m_1}^{r_{|m_1|}} \otimes \cdots \otimes v_{m_n}^{r_{|m_n|}}.$$

where Φ_w denotes $\Phi(T_w)$.

Let $I = (I_1, \dots, I_m) \in \mathbf{P}(\mathbf{n})$ and $A \subseteq \mathbf{n}$, we will denote $\Phi(E_I)$ and $\Phi(F_A)$ by \mathbf{E}_I and \mathbf{F}_A respectively. We know by [8] that \mathbf{E}_I acts over $V^{\otimes n}$ as follows

$$(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})\mathbf{E}_I = \begin{cases} 0 & \text{if there exist } i, j, k \text{ such that } i, j \in I_k \text{ y } r_j \neq r_j, \\ v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & \text{otherwise} \end{cases} \quad (36)$$

In the same fashion, it is not difficult to prove that

$$(v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n})\mathbf{F}_A = \begin{cases} 0 & \text{if there exist } i \in A \text{ such that } r_i \neq 0 \\ v_{i_1}^{r_1} \otimes \cdots \otimes v_{i_n}^{r_n} & \text{otherwise} \end{cases} \quad (37)$$

Now, we have all the necessary to prove the main result of this section

Theorem 2. *The set \mathcal{B}_n is a basis of $\mathcal{E}_n^{\mathbf{B}}$. In particular, the dimension of $\mathcal{E}_n^{\mathbf{B}}$ is $b_{n+1}2^n n!$*

Proof. We only have to prove that \mathcal{B}_n is a linear independent set, since it was already proven in Proposition 4 that it spans \mathcal{E}_n^B . Let be $I = (I_0, I_1, \dots, I_m)$ a element in $P(\mathbf{n}_0)$ considering the single blocks in its expression. Without lose of generality set I_0 as the block that contains 0, then, we define $v^I \in V^{\otimes n}$ as follows

$$v^I := v_1^{r_1} \otimes \dots \otimes v_n^{r_n}, \quad \text{with } r_i = k \text{ if } i \in I_k$$

Suppose now that

$$\sum_{J \in P(\mathbf{n}_0), w \in W_n} \lambda_{J,w} E F_J T_w = 0 \quad (38)$$

Then, given $I \in P(\mathbf{n}_0)$, if we apply Φ and evaluate v^I in (38), we will obtain

$$\sum_{J \in P(\mathbf{n}_0), w \in W_n} \lambda_{J,w} (v^I) (\mathbf{E} \mathbf{F}_J \Phi_w) = 0$$

thus, using (36) and (37) we have

$$\begin{aligned} \sum_{w \in W_n} \lambda_{I,w} (v^I) \Phi_w &= 0 \\ \sum_{w \in W_n} \lambda_{I,w} (v_1^{r_1} \otimes \dots \otimes v_n^{r_n}) \Phi_w &= 0 \\ \sum \lambda_{I,w} v_{m_1}^{r_{|m_1|}} \otimes \dots \otimes v_{m_n}^{r_{|m_n|}} &= 0 \end{aligned} \quad (39)$$

with (m_1, \dots, m_n) running in X_n^n . But, this elements are L.I in $V^{\otimes n}$, then $\lambda_{I,w} = 0$ for all $w \in W_n$. Finally as I was picked arbitrarily the result follows. \square

Corollary 2. *The representation Φ is faithful.*

Remark 5. *Since $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$, it follows that $\mathcal{E}_n^B \subseteq \mathcal{E}_{n+1}^B$, for all $n \geq 1$. Thus, by taking $\mathcal{E}_0^B := \mathbb{K}$, we have the following tower of algebras.*

$$\mathcal{E}_0^B \subseteq \mathcal{E}_1^B \subseteq \dots \subseteq \mathcal{E}_n^B \subseteq \mathcal{E}_{n+1}^B \subseteq \dots \quad (40)$$

Remark 6. Recall that in the original definition of Φ in [8], V is considered as the vector space with basis $\mathcal{B} = \{v_i^r; i \in X_n, 0 \leq r \leq d-1\}$. Thus, the condition $d = n+1$ it is essential for proving Theorem 2, in fact, if we suppose $d \leq n$, we could obtain a sum of linear dependent elements in Eq. (39). Moreover, to get a sum of linear independent elements, it is enough to consider $d \geq n+1$, which is contained in the next result.

Corollary 3. *Suppose that $d \geq n+1$. Then the homomorphism $\varphi_n : \mathcal{E}_n^B \mapsto Y_{d,n}^B$ defined in Proposition 1 is an embedding.*

Proof. It is enough to prove that the set $A = \varphi_n(\mathcal{B}_n)$ is linear independent in $Y_{d,n}^B$. Now, we know that Φ is faithful ([8, Corollary]), and we proved in Theorem 2 that $\Phi(A)$ is L.I, then, the result follows. \square

Corollary 4. *The set $\mathcal{C}_n = \{E F_I(\mathbf{m}_1 \dots \mathbf{m}_n) \mid I \in P(\mathbf{n}_0), \mathbf{m}_i \in M_i\}$ is also a basis for \mathcal{E}_n^B .*

Proof. We can prove that \mathcal{C}_n spans \mathcal{E}_n^B analogously to Proposition 4, but this time using the relations given in Lemma 2. The linear independence is guaranteed by cardinality. \square

5. MARKOV TRACE IN \mathcal{E}_n^B

In this section we prove that \mathcal{E}_n^B supports a Markov Trace. For that, we use the method of relative traces, cf.[2, 6], which consists in construct a family of linear maps $\vartheta_n : \mathcal{E}_n^B \rightarrow \mathcal{E}_{n-1}^B$, which gives step by step the desired Markov properties. Specifically, these properties are guaranteed by three key results, in our case these are Lemmas 8, Lemma 11 and (ii) Lemma 9 (ii), which are essential to prove that the trace defined by $\text{tr}_n = \vartheta_1 \circ \dots \circ \vartheta_n$ is a Markov trace (Theorem 3).

5.1. From now on, we fix the parameters $x, w, y, z \in \mathbb{K}$ and we consider $w \in \mathcal{C}_n$ expressed on the form $w = m_1 \cdots m_{n-1} EF_I$ (which is possible by Corollary 1). Let $\mathbb{L} := \mathbb{K}(x, w, y, z)$, then, when it is needed, we will consider x, w, y, z as variables, and work with the algebra $\mathcal{E}_n^B \otimes \mathbb{L}$, which, for simplicity, will be denoted by \mathcal{E}_n^B again.

Definition 3. We set $\vartheta_1(B_1) = y$, $\vartheta_1(F_1) = x$ and $\vartheta_1(B_1 F_1) = w$. For $n \geq 2$, we define the linear map from \mathcal{E}_n^B to \mathcal{E}_{n-1} on the basis \mathcal{C}_n as follows:

$$\vartheta_n(m_1 \cdots m_{n-1} m_n EF_I) = \begin{cases} m_1 \cdots m_{n-1} EF_I & \text{for } m_n = 1, n \notin \text{Supp}(I), \\ x m_1 \cdots m_{n-1} EF_{I \setminus n} & \text{for } m_n = 1, n \in \text{Supp}(I), \\ z m_1 \cdots m_{n-1} T_{n-1, k}^\pm EF_{\tau_{n, k}(I)} & \text{for } m_n = T_{n, k}^\pm; k < n, \\ y m_1 \cdots m_{n-1} EF_I & \text{for } m_n = B_n, n \notin \text{Supp}(I), \\ w m_1 \cdots m_{n-1} EF_{I \setminus n} & \text{for } m_n = B_n, n \in \text{Supp}(I). \end{cases} \quad (41)$$

where $\tau_{n, k}(I)$ denotes the partition $(I * \{n, k\}) \setminus n$.

We begin proving some partitions properties, which will use frequently in the sequel

Lemma 5. Let $\sigma \in S_n$ and $I \in \mathcal{P}(\mathbf{n}_0)$, then we have that

- (i) $\sigma(I \setminus \{k\}) = \sigma(I) \setminus \{\sigma(k)\}$, for some $k \in \text{Supp}(I)$.
- (ii) $\sigma(I * \{j, k\}) = \sigma(I) * \{\sigma(j), \sigma(k)\}$, for some $k, j \in \mathbf{n}$.
- (iii) $\sigma(\tau_{n, k}(I)) = \tau_{\sigma(n), \sigma(k)}(\sigma(I))$ for some $k < n$.

Proof. Let $I = (I_1, \dots, I_m) \in \mathcal{P}(\mathbf{n}_0)$, and suppose without lose of generality that I_1 contains k . Then we have that $I \setminus \{k\} = (I'_1, I_2, \dots, I_m)$ where $I'_1 = I_1 \setminus \{k\}$. Therefore

$$\begin{aligned} \sigma(I \setminus \{k\}) &= (\sigma(I'_1), \sigma(I_2), \dots, \sigma(I_m)) \\ &= (\sigma(I_1) \setminus \{\sigma(k)\}, \sigma(I_2), \dots, \sigma(I_m)) = \sigma(I) \setminus \{\sigma(k)\} \end{aligned}$$

and we have (i). For (ii) we only prove the case when $j, k \in \text{Supp}(I)$, and these are in different blocks. Let I_1 y I_2 the blocks of I that contains j and k respectively. Then, we have

$$I * \{j, k\} = (I_1 \cup I_2, I_3, \dots, I_m)$$

therefore

$$\begin{aligned} \sigma(I * \{n, k\}) &= (\sigma(I_1) \cup \sigma(I_2), \sigma(I_3), \dots, \sigma(I_m)) \\ &= (\sigma(I_1), \sigma(I_2), \sigma(I_3), \dots, \sigma(I_m)) * \{\sigma(n), \sigma(k)\} = \sigma(I) * \{\sigma(j), \sigma(k)\} \end{aligned}$$

Finally, (iii) is a consequence of (i) and (ii). \square

We would like to prove that (41) holds by considering $v \in \mathcal{E}_{n-1}^B$ instead $m_1 \cdots m_{n-1}$ in the formula. Having this in mind, we introduce some notation and we prove a technical lemma.

Let $j > k \in \mathbf{n}$ we define the element $\sigma_{j, k} \in S_n$ by

$$\sigma_{j, k} = s_{j-1} \cdots s_k$$

where the s_i denote the transposition $(i, i+1)$. Note that

$$\sigma_{j, k}(i) = \begin{cases} j & \text{if } i = k \\ i-1 & \text{if } k < i \leq j \\ i & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_{j, k}^{-1}(i) = \begin{cases} k & \text{if } i = j \\ i+1 & \text{if } k \leq i < j \\ i & \text{otherwise} \end{cases} \quad (42)$$

Lemma 6. For $J = (J_1, \dots, J_r) \in \mathcal{P}(\mathbf{n}_0 \setminus \{n\})$ and $I = (I_1, \dots, I_s) \in \mathcal{P}(\mathbf{n}_0)$ the following equality holds.

$$(\sigma_{n, k}^{-1}(J) * I) * \{n, k\} \setminus n = \sigma_{n-1, k}^{-1}(J) * ((I * \{n, k\}) \setminus n) \quad (43)$$

Proof. We only prove the case when $n, k \in \text{Supp}(I)$, and these are in different blocks of I , since the other cases can be verified analogously. Let $J = (J_1, \dots, J_r) \in \mathcal{P}(\mathbf{n}_0 \setminus \{n\})$ and $I = (I_1, \dots, I_s) \in \mathcal{P}(\mathbf{n}_0)$, without lose generality, we can suppose that I_1 and I_2 are the blocks of I that contain k and n respectively. We proceed, distinguish cases.

CASE: $n - 1 \notin \text{Supp}(J)$

First, note that in this case the partitions $\sigma_{n,k}^{-1}(J)$ and $\sigma_{n-1,k}^{-1}(J)$ are the same, and it will be denoted by $A = (A_1, \dots, A_r)$. Moreover, we have that $n, k \notin \text{Supp}(A)$ by (42) and the fact that $n - 1 \notin \text{Supp}(J)$. Then, in these case (43) holds directly, since the operations $*\{n, k\}$ and $\setminus n$ just have influence over I .

CASE: $n - 1 \in \text{Supp}(J)$

This time $\sigma_{n,k}^{-1}(J)$ and $\sigma_{n-1,k}^{-1}(J)$ are different, we will denote these by $A = (A_1, \dots, A_r)$ and $B = (B_1, \dots, B_r)$ respectively. Note that, $n \in \text{Supp}(A)$, $k \in \text{Supp}(B)$, $k \notin \text{Supp}(A)$ and $n \notin \text{Supp}(B)$. Moreover, if A_1 and B_1 are the blocks of A and B that contain n and k respectively, we have that $A_i = B_i$ for $2 \leq i \leq r$, and $A_1 \setminus \{n\} = B_1 \setminus \{k\}$.

Let I_3, \dots, I_t , with $t \leq s$, the blocks of I that have intersection with $A_1 \setminus \{n\} = B_1 \setminus \{k\}$, and I_{t+1}, \dots, I_s those that don't have. We define the partitions

$$A' = (A_2, \dots, A_r), \quad B' = (B_2, \dots, B_r), \quad \text{and} \quad I' = (I_{t+1}, \dots, I_s)$$

Then, for one side we have

$$\begin{aligned} B * ((I * \{n, k\}) \setminus \{n\}) &= B * ((I_1 \cup I_2) \setminus n, I_3, \dots, I_t, I') \\ &= B * (I_1 \cup (I_2 \setminus \{n\}), I_3, \dots, I_t, I') \\ &= (B_1 \cup I_1 \cup (I_2 \setminus \{n\}) \cup I_3 \cup \dots \cup I_s, B' * I') \\ &= ((B_1 \setminus \{k\}) \cup I_1 \cup (I_2 \setminus \{n\}) \cup I_3 \cup \dots \cup I_s, B' * I') \end{aligned}$$

On the other hand we have

$$\begin{aligned} ((A * I) * \{n, k\}) &= ((A_1 \cup I_2 \cup I_3 \cup \dots \cup I_t, I_1, A' * I') * \{n, k\}) \setminus n \\ &= ((A_1 \cup I_1 \cup I_2 \cup I_3 \cup \dots \cup I_t) \setminus n, A' * I') \\ &= ((A_1 \setminus \{n\}) \cup I_1 \cup (I_2 \setminus \{n\}) \cup I_3 \cup \dots \cup I_t, A' * I') \end{aligned}$$

since $A' = B'$ and $A_1 \setminus \{n\} = B_1 \setminus \{k\}$ the result follows. \square

Lemma 7. For every $v \in \mathcal{E}_{n-1}^B$ we have

$$\vartheta_n(v\mathfrak{m}_n EF_I) = \begin{cases} vEF_I & \text{for } \mathfrak{m}_n = 1, n \notin \text{Supp}(I), \\ xvEF_{I \setminus n} & \text{for } \mathfrak{m}_n = 1, n \in \text{Supp}(I), \\ zv\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(I)} & \text{for } \mathfrak{m}_n = \mathbb{T}_{n,k}^\pm; k < n, \\ yvEF_I & \text{for } \mathfrak{m}_n = B_n, n \notin \text{Supp}(I), \\ wvEF_{I \setminus n} & \text{for } \mathfrak{m}_n = B_n, n \in \text{Supp}(I). \end{cases}$$

Proof. By the linearity of the trace is enough prove the statement for $v \in \mathcal{C}_{n-1}$. The cases when $\mathfrak{m}_n = 1$ can be proven easily.

For case $\mathfrak{m}_n = \mathbb{T}_{n,k}^\pm$ with $k < n$, we have

$$\begin{aligned}
\vartheta_n(v\mathbb{T}_{n,k}^\pm EF_I) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_J \mathbb{T}_{n,k}^\pm EF_I) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n,k}^\pm EF_{\sigma_{n,k}^{-1}(J)} EF_I) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n,k}^\pm EF_{\sigma_{n,k}^{-1}(J)*I}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(\sigma_{n,k}^{-1}(J)*I)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbf{z} v\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(I)} &= \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_J \mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(I)} \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^\pm EF_{\sigma_{n-1,k}^{-1}(J)} EF_{\tau_{n,k}(I)} \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^\pm EF_{\sigma_{n-1,k}^{-1}(J)*\tau_{n,k}(I)}
\end{aligned}$$

Then the result follows by Lemma 6.

Finally, we suppose that $\mathfrak{m}_n = B_n$. We only prove the case when $n \in \text{Supp}(I)$, since the opposite case can be verified by an analogous way. Then, we have

$$\begin{aligned}
\vartheta_n(vB_n EF_I) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_J B_n EF_I) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n EF_J EF_I) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n EF_{I*J}) \\
&= \mathbf{w} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_{(I*J)\setminus n} \\
&= \mathbf{w} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_{(I\setminus n)*J} \\
&= \mathbf{w} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} EF_J EF_{I\setminus n} = \mathbf{w} v EF_{I\setminus n}
\end{aligned}$$

□

The following lemmas contain several computations analogous to the proved in [2, Section 5] for the bt-algebra. Therefore, although we work with a different quadratic relation, we will omit some computations in the following proofs, since these can be obtained by passing through the automorphism induced by the change of generators given in Remark 2.

Lemma 8. For all $X, Z \in \mathcal{E}_{n-1}^B$ and $Y \in \mathcal{E}_n^B$, we have:

- (i) $\vartheta_n(YZ) = \vartheta_n(Y)Z$
- (ii) $\vartheta_n(XY) = X\vartheta_n(Y)$

Proof. For proving claim (i) notice that, due to the linearity of ϑ_n , we can suppose that Z is a defining generator of \mathcal{E}_{n-1}^B and $Y = \mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n EF_I$, with $\mathfrak{m}_i \in M_i$ and $I \in P(\mathfrak{n}_0)$. To prove the claim we shall distinguish the Y 's according to the possibilities of \mathfrak{m}_n , and if n belongs to $\text{Supp}(I)$ or not.

First, note that for $Z = E_i, F_j$ with $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$, claim (i) holds easily for any choice of \mathfrak{m}_n . Also, when $\mathfrak{m}_n = 1$ and $n \notin \text{Supp}(I)$, since ϑ_n acts like the identity. Now, we proceed to study the remaining cases.

CASE: $\mathfrak{m}_n = 1, n \in \text{Supp}(I)$

If we consider $Z = T_j, E_j$ the result follows by [2, Lemma 2]. And for $Z = B_1, F_1$ the results follows by Lemma 7 and the fact that B_1 commutes with EF_I .

CASE: $\mathfrak{m}_n = B_n, n \in \text{Supp}(I)$

First suppose that $Z = T_j$ for $j \in \{1, \dots, n-2\}$

$$\begin{aligned} \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n E F_I T_j) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n T_j E F_{s_j(I)}) \\ &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_j B_n E F_{s_j(I)}) \\ &= w \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_j E F_{s_j(I) \setminus n} \end{aligned}$$

On the other hand, $\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n E F_I) T_j = w \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_j E F_{s_j(I) \setminus n}$. Thus, since s_j doesn't act over n , we have that $s_j(I) \setminus n = s_j(I \setminus n)$, and the result follows.

For $Z = B_1$, we have

$$\begin{aligned} \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n E F_I B_1) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_n B_1 E F_I) \\ &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 B_n E F_I + (u - u^{-1}) \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \lambda_n E F_I) \\ &= w \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 E F_{I \setminus n} + \vartheta_n((u - u^{-1}) \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \lambda_n E F_I) \\ &= w \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} E F_{I \setminus n} B_1 + (u - u^{-1}) \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \lambda_n E F_I) \end{aligned}$$

by using ii) Lemma 2, where $\lambda_n = [B_1 T_1^{-1} \cdots T_{n-2}^{-1} T_{n-1} \cdots T_1 B_1 E_{1,k} - T_1^{-1} \cdots T_{n-2}^{-1} T_{n-1} \cdots T_1 B_1^2 E_{1,k}]$. Then, it is enough to prove that $A = \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \lambda_n E F_I) = 0$. In fact, we have

$$\begin{aligned} A &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} [B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^- E_{1,k} - T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^+ B_1^2 E_{1,k}] E F_I) \\ &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^- E_{1,k} E F_I) - \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^+ B_1^2 E_{1,k} E F_I) \\ &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^- E F_{I * \{1,k\}}) - \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^+ E F_{I * \{1,k\}}) - \\ &\quad (v - v^{-1}) \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n,1}^- E F_{I * \{0,1,k\}}) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n-1,1}^- E F_{\tau_{n,1}(I * \{1,k\})} - z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n-1,1}^+ E F_{\tau_{n,1}(I * \{1,k\})} - \\ &\quad z(v - v^{-1}) \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n-1,1}^- E F_{\tau_{n,1}(I * \{0,1,k\})} \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1^2 E F_{\tau_{n,1}(I * \{1,k\})} - z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} E F_{\tau_{n,1}(I * \{1,k\})} - \\ &\quad z(v - v^{-1}) \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 E F_{\tau_{n,1}(I * \{0,1,k\})} \end{aligned}$$

Finally, expanding B_1^2 , we obtain that $\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \lambda_n E F_I) = 0$, since $\tau_{n,1}(I * \{1,k\}) * \{0,1\} = \tau_{n,1}(I * \{0,1,k\})$. For the case $\mathfrak{m}_n = B_n, n \notin \text{Supp}(I)$, we can proceed analogously (we only have to put y instead w , and omit the operation $\setminus n$ in the partition).

CASE: $\mathfrak{m}_n = \mathbb{T}_{n,k}^+$

For $Z = T_j$ with $j \in \{1, \dots, n-2\}$ the result follows analogously as in [2, Lemma 1] using the relations of Lemma 2. Suppose now, that $Z = B_1$, if $k > 1$, then B_1 commutes with $\mathbb{T}_{n,k}^+$, therefore the result follows easily. If $k = 1$, we have

$$\begin{aligned} \vartheta_n(YZ) = \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_{n,k}^+ E F_I B_1) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_{n,1}^+ B_1 E F_I) \\ &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_{n,1}^- E F_I) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_{n-1,1}^- E F_{\tau_{n,1}(I)} \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_{n-1,1}^+ E F_{\tau_{n,1}(I)} B_1 = \vartheta_n(Y)Z \end{aligned}$$

CASE: $\mathfrak{m}_n = \mathbb{T}_{n,k}^-$

For $Z = T_j$ with $j \in \{1, \dots, n-2\}$, the proof follows analogously as in [2, Lemma 1], since the formula i) of Lemma 2 coincides with (22) of [2].

Finally, for $Z = B_1$ we have

$$\begin{aligned}
\vartheta_n(YZ) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n,k}^- EF_I B_1) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n,k}^- B_1 EF_I) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 \mathbb{T}_{n,k}^- EF_I) + (\mathfrak{u} - \mathfrak{u}^{-1})[\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n,1}^- E_{1,k} EF_I) - \\
&\quad \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n,1}^- B_1 E_{1,k} EF_I)] \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 \mathbb{T}_{n,k}^- EF_I) + (\mathfrak{u} - \mathfrak{u}^{-1})[\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n,1}^- EF_{I*\{1,k\}}) - \\
&\quad \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n,1}^+ EF_{I*\{1,k\}})] - (\mathfrak{v} - \mathfrak{v}^{-1})\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n,1}^- EF_{I*\{0,1,k\}})] \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 \mathbb{T}_{n-1,k}^- EF_{\tau_{n,k}(I)} + \mathfrak{z}(\mathfrak{u} - \mathfrak{u}^{-1})[\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^- EF_{\tau_{n,1}(I*\{1,k\})} - \\
&\quad \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^+ EF_{\tau_{n,1}(I*\{1,k\})}] - (\mathfrak{v} - \mathfrak{v}^{-1})\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^- EF_{\tau_{n,1}(I*\{0,1,k\})}]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\vartheta_n(Y)Z &= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^- EF_{\tau_{n,k}(I)} B_1 \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^- B_1 EF_{\tau_{n,k}(I)} \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \mathbb{T}_{n-1,k}^- B_1 EF_{\tau_{n,k}(I)} \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 \mathbb{T}_{n-1,k}^- EF_{\tau_{n,k}(I)} + \mathfrak{z}(\mathfrak{u} - \mathfrak{u}^{-1})[\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} B_1 T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^- EF_{\tau_{n,k}(I)*\{1,k\}} - \\
&\quad \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^+ EF_{\tau_{n,k}(I)*\{1,k\}}] - (\mathfrak{v} - \mathfrak{v}^{-1})\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} T_1^{-1} \cdots T_{k-2}^{-1} \mathbb{T}_{n-1,1}^- EF_{\tau_{n,k}(I)*\{0,1,k\}}]
\end{aligned}$$

clearly we have that $\tau_{n,k}(I) * \{1, k\} = \tau_{n,1}(I * \{1, k\})$ and $\tau_{n,1}(I * \{0, 1, k\}) = \tau_{n,k}(I) * \{0, 1, k\}$, then, the result follows.

Finally (ii) is a direct consequence of Lemma 7, since $X\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1} \in \mathcal{E}_{n-1}^B$. \square

Corollary 5. For all $X, Z \in \mathcal{E}_{n-1}^B$ and $Y \in \mathcal{E}_n^B$, we have:

$$\vartheta_n(XYZ) = X\vartheta_n(Y)Z.$$

Proof. The proof is straightforward using the previous lemmas. \square

Lemma 9. For $n \geq 2$, $X \in \mathcal{E}_{n-1}^B$ and $Y \in \mathcal{E}_n^B$, we have

- (i) $\vartheta_n(E_{n-1}XT_{n-1}) = \vartheta_n(T_{n-1}XE_{n-1})$
- (ii) $\vartheta_{n-1}(\vartheta_n(E_{n-1}Y)) = \vartheta_{n-1}(\vartheta_n(YE_{n-1}))$

Proof. As always, by linearity of the trace, we can consider X and Y in \mathcal{C}_{n-1} and \mathcal{C}_n respectively. Let $X = \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathfrak{m}_{n-1}EF_I$, with $I \in \mathcal{P}(\mathfrak{n}_0 \setminus \{n\})$, for proving (i) we will distinguish cases depending of the value of \mathfrak{m}_{n-1} .

CASE: $\mathfrak{m}_{n-1} = 1$. For one side, we have

$$\begin{aligned}
\vartheta_n(E_{n-1}XT_{n-1}) &= \vartheta_n(E_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_IT_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}EF_{s_{n-1}(I)}E_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}EF_{s_{n-1}(I)*\{n-1,n\}}) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{\tau_{n,n-1}(s_{n-1}(I)*\{n-1,n\})}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_n(T_{n-1}XE_{n-1}) &= \vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_I E_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}EF_{I*\{n-1,n\}}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{\tau_{n,n-1}(I*\{n-1,n\})}
\end{aligned}$$

Now, if $n-1 \notin \text{Supp}(I)$ the equality is clear, since $s_{n-1}(I) = I$. On the other hand, if $n-1 \in \text{Supp}(I)$, it is not difficult to prove that $s_{n-1}(I) * \{n-1, n\} = I * \{n-1, n\}$.

CASE: If $\mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$, we have

$$\begin{aligned}
\vartheta_n(E_{n-1}XT_{n-1}) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm E_{n,k}T_{n-1}EF_{s_{n-1}(I)}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm T_{n-1}EF_{s_{n-1}(I)*\{n-1,k\}}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,n-1}(s_{n-1}(I)*\{n-1,k\})}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_n(T_{n-1}XE_{n-1}) &= \vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm EF_I E_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,k}^\pm EF_{I*\{n-1,n\}}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(I*\{n-1,n\})}
\end{aligned}$$

and, it is easy to verify that the partitions from both cases are equal.

CASE: If $\mathfrak{m}_{n-1} = B_{n-1}$, we have

$$\begin{aligned}
\vartheta_n(E_{n-1}XT_{n-1}) &= \vartheta_n(E_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_I T_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}T_{n-1}E_{n-1}EF_{s_{n-1}(I)}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}T_{n-1}EF_{s_{n-1}(I)*\{n-1,n\}}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_{\tau_{n,n-1}(s_{n-1}(I)*\{n-1,n\})}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_n(T_{n-1}XE_{n-1}) &= \vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_I E_{n-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}B_{n-1}EF_{I*\{n-1,n\}}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,n-1}^- EF_{I*\{n-1,n\}}) \\
&= \mathbf{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_{\tau_{n,n-1}(I*\{n-1,n\})}
\end{aligned}$$

and we know by the first case that the partitions involved are equal, then we have already proved (i).

For proving (ii) we need more cases, since we have to apply two levels of the relative trace, then the result depends from the values of \mathfrak{m}_{n-1} and \mathfrak{m}_n of $Y = \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathfrak{m}_{n-1}\mathfrak{m}_n EF_I \in \mathcal{C}_n$.

First note that for the cases $\mathfrak{m}_n = 1, \mathfrak{m}_{n-1} = 1$; $\mathfrak{m}_n = 1, \mathfrak{m}_{n-1} = B_{n-1}$; $\mathfrak{m}_n = B_n, \mathfrak{m}_{n-1} = 1$ and $\mathfrak{m}_n = B_n, \mathfrak{m}_{n-1} = B_{n-1}$, the result follows directly, since E_{n-1} commute with Y in each case, then we only have to analyze five cases.

CASE: If $\mathfrak{m}_n = 1$ and $\mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$.

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(YE_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^\pm EF_I E_{n-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^\pm EF_{I*\{n-1,n\}})) \\
&= \times \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^\pm EF_{(I*\{n-1,n\}) \setminus n}) \\
&= \times \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(I_1)}
\end{aligned}$$

where $I_1 = (I * \{n-1, n\}) \setminus n$. On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(E_{n-1}Y)) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} E_{n-1} \mathbb{T}_{n-1,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^\pm EF_{I*\{n,k\}})) \\
&= \times \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^\pm EF_{(I*\{n,k\}) \setminus n}) \\
&= \times \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(I_2)}
\end{aligned}$$

where $I_2 = I * \{n, k\} \setminus n$. Further, we know by [2, Lemma 2] that $\tau_{n-1,k}(I_1) = \tau_{n-1,k}(I_2)$. Note that for case $\mathfrak{m}_n = B_n$ and $\mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$ we have an analogous proof, since $E_{n,k}$ commutes with B_n , indeed, the only difference with this case is that when we apply the trace at level n appear the parameter w instead x .

CASE: If $\mathfrak{m}_n = \mathbb{T}_{n,k}^\pm$ and $\mathfrak{m}_{n-1} = 1$.

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(YE_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n,k}^\pm EF_I E_{n-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n,k}^\pm EF_{I*\{n-1,n\}})) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_{I_1})
\end{aligned}$$

where $I_1 = \tau_{n,k}(I * \{n-1, n\})$. On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(E_{n-1}Y)) &= \vartheta_{n-1}(\vartheta_n(E_{n-1} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n,k}^\pm EF_{I*\{n,k\}})) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_{I_2})
\end{aligned}$$

where $I_2 = \tau_{n,k}(I * \{n, k\})$. First note that when $k = n-1$ the result follows directly, and if $k < n-1$ we obtain

$$\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_{I_i}) = \mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(I_i)}$$

Again by [2, Lemma 2] we have that $\tau_{n-1,k}(I_1) = \tau_{n-1,k}(I_2)$ and the result follows.

CASE: If $\mathfrak{m}_n = \mathbb{T}_{n,k}^\pm$ and $\mathfrak{m}_{n-1} = B_{n-1}$.

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(YE_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} \mathbb{T}_{n,k}^\pm EF_I E_{n-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} \mathbb{T}_{n,k}^\pm EF_{I*\{n-1,n\}})) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(B_{n-1} \mathbb{T}_{n-1,k}^\pm EF_{I_1})
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(E_{n-1}Y)) &= \vartheta_{n-1}(\vartheta_n(E_{n-1} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} \mathbb{T}_{n,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} \mathbb{T}_{n,k}^\pm E_{n,k} EF_I)) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(B_{n-1} \mathbb{T}_{n-1,k}^\pm EF_{I_2})
\end{aligned}$$

where $I_1 = \tau_{n,k}(I * \{n-1, n\})$ and $I_2 = \tau_{n,k}(I * \{n, k\})$. Now, note that when $k = n-1$ the result is direct, then we can suppose that $k < n-1$, thus we have that

$$B_{n-1}\mathbb{T}_{n-1,k}^{\pm} = B_{n-1}T_{n-2}\mathbb{T}_{n-2,k}^{\pm} = \mathbb{T}_{n-1,n-2}^{-}\mathbb{T}_{n-2,k}^{\pm} = T_{n-2}B_{n-2}\mathbb{T}_{n-2,k}^{\pm}$$

Using this we obtain for one side

$$\begin{aligned}\vartheta_{n-1}(B_{n-1}\mathbb{T}_{n-1,k}^{\pm}EF_{I_j}) &= \vartheta_{n-1}(\mathbb{T}_{n-1,n-2}^{-}\mathbb{T}_{n-2,k}^{\pm}EF_{I_j}) \\ &= \vartheta_{n-1}(\mathbb{T}_{n-1,n-2}^{-}EF_{\sigma(I_j)})\mathbb{T}_{n-2,k}^{\pm} \\ &= z B_{n-2}EF_{\tau_{n-1,n-2}(\sigma(I_j))}\mathbb{T}_{n-2,k}^{\pm}\end{aligned}$$

where $\sigma = \sigma_{n-2,k}$. Let see that the partitions are equal. let be A_1, A_2, A_3 the blocks of I that contains $k, n-1$ and n respectively, consider I' as the partition that result by removing the blocks A_1, A_2 and A_3 from I . Then, we have

$$\begin{aligned}I_1 &= (I * \{n, n-1\}) * \{n, k\} \setminus n = ((I * \{n, n-1, k\}) \setminus n = I' * (A_1 \cup A_2 \cup A'_3)) \\ I_2 &= ((I * \{n, k\}) * \{n, k\}) \setminus n = ((I * \{n, k\}) \setminus n = I' * (A_1 \cup A'_3, A_2))\end{aligned}$$

where $A'_3 = A_3 \setminus \{n\}$. Therefore

$$\begin{aligned}\sigma(I_1) &= \sigma(I') * (\sigma(A_1) \cup \sigma(A_2) \cup \sigma(A'_3)) \\ \sigma(I_2) &= \sigma(I') * (\sigma(A_1) \cup \sigma(A'_3), \sigma(A_2))\end{aligned}$$

now, note that $\sigma(k) = n-2$ and $\sigma(n-1) = n-1$, then we have

$$\begin{aligned}\tau_{n-1,n-2}(\sigma(I_1)) &= (\sigma(I_2) * \{n-1, n-2\}) \setminus n-1 = \sigma(I') * (\sigma(A_1) \cup \sigma(A'_2) \cup \sigma(A'_3)) \\ \tau_{n-1,n-2}(\sigma(I_2)) &= (\sigma(I_1) * \{n-1, n-2\}) \setminus n-1 = \sigma(I') * (\sigma(A_1) \cup \sigma(A'_3) \cup \sigma(A'_2))\end{aligned}$$

where $A_2 = A_2 \setminus \{n-1\}$.

CASE: If $\mathfrak{m}_n = \mathbb{T}_{n,k}^{\pm}$ and $\mathfrak{m}_{n-1} = \mathbb{T}_{n-1,j}^{\pm}$. Similarly as the last case, we have

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(YE_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n,k}^{\pm}EF_{I * \{n-1,n\}})) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}EF_{I_1})\end{aligned}$$

On the other hand

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(E_{n-1}Y)) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}E_{n-1}\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n,k}^{\pm}EF_I)) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}E_{a,b}EF_I)) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}EF_{I_2})\end{aligned}$$

where $I_1 = \tau_{n,k}(I * \{n-1, n\})$, $I_2 = \tau_{n,k}(I * \{a, b\})$ and

$$\{a, b\} = \begin{cases} \{k, j\} & \text{if } j < k \\ \{k, j+1\} & \text{if } j \geq k \end{cases}$$

• SUBCASE $k < n-1$: First, if $j = n-2$ implies that $j \geq k$, therefore $\{a, b\} = \{j+1, k\} = \{n-1, k\}$, thus $I_1 = I_2$, and the result follows. Then, we can suppose $j < n-2$, and we obtain

$$\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm} = T_{n-2}T_{n-3}\mathbb{T}_{n-3,j}^{\pm}T_{n-2}\mathbb{T}_{n-2,k}^{\pm} = T_{n-2}T_{n-3}T_{n-2}\mathbb{T}_{n-3,j}^{\pm}\mathbb{T}_{n-2,k}^{\pm} = T_{n-3}T_{n-2}\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-2,k}^{\pm}$$

thus

$$\begin{aligned}\vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}EF_{I_1})i &= T_{n-3}\vartheta_{n-1}(T_{n-2}EF_{I'_i})\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-2,k}^{\pm} \\ &= z T_{n-3}EF_{\tau_{n-1,n-2}(I'_i)}\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-2,k}^{\pm}\end{aligned}$$

where $I'_i = \sigma(I_i)$, with $\sigma = \sigma_{n-2,j}\sigma_{n-2,k}$, for $i = 1, 2$. And, it is known that $\tau_{n-1,n-2}(I'_1) = \tau_{n-1,n-2}(I'_2)$ by [2, Lemma 2].

• **SUBCASE $k = n - 1$:** We have that $\mathbb{T}_{n-1,k}^\pm = 1$ or B_{n-1} , for the first case the result is direct. Then, suppose $\mathbb{T}_{n-1,k}^\pm = B_{n-1}$, we proceed with the positive case first, that is $\mathbb{m}_{n-1} = \mathbb{T}_{n-1,j}^+$. Note that

$$\begin{aligned} \mathbb{T}_{n-1,j}^+ B_{n-1} &= T_{n-2} \cdots T_j T_{n-2} \cdots T_1 B_1 T_1^{-1} \cdots T_{n-2}^{-1} \\ &= T_{n-2}^2 \cdots T_1 B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n-2,j}^+ \\ &= T_{n-3} \cdots T_1 B_1 T_1^{-1} \cdots T_{n-2}^{-1} \mathbb{T}_{n-2,j}^+ + (\mathbf{u} - \mathbf{u}^{-1}) B_{n-1} E_{n-2} \mathbb{T}_{n-2,j}^+ \\ &= B_{n-2} T_{n-2} \mathbb{T}_{n-2,j}^+ - (\mathbf{u} - \mathbf{u}^{-1}) B_{n-2} E_{n-2} \mathbb{T}_{n-2,j}^+ + (\mathbf{u} - \mathbf{u}^{-1}) B_{n-1} E_{n-2} \mathbb{T}_{n-2,j}^+ \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^+ B_{n-1} E F_{I_i}) &= \vartheta_{n-1}(B_{n-2} T_{n-2} \mathbb{T}_{n-2,j}^+ E F_{I_i}) - (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-2} E_{n-2} \mathbb{T}_{n-2,j}^+ E F_{I_i}) + \\ &\quad (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-1} E_{n-2} \mathbb{T}_{n-2,j}^+ E F_{I_i}) \\ &= B_{n-2} \vartheta_{n-1}(T_{n-2} E F_{I'_i}) \mathbb{T}_{n-2,j}^+ - (\mathbf{u} - \mathbf{u}^{-1}) B_{n-2} \vartheta_{n-1}(E F_{I'_i * \{n-1, n-2\}}) \mathbb{T}_{n-2,j}^+ + \\ &\quad (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-1} E F_{I'_i * \{n-1, n-2\}}) \mathbb{T}_{n-2,j}^+ \\ &= \mathbf{z} B_{n-2} E F_{\tau_{n-1,n-2}(I'_i)} \mathbb{T}_{n-2,j}^+ - \mathbf{x} (\mathbf{u} - \mathbf{u}^{-1}) B_{n-2} E F_{(I'_i * \{n-1, n-2\}) \setminus n-1} \mathbb{T}_{n-2,j}^+ + \\ &\quad \mathbf{w} (\mathbf{u} - \mathbf{u}^{-1}) B_{n-1} E F_{(I'_i * \{n-1, n-2\}) \setminus n-1}) \mathbb{T}_{n-2,j}^+ \end{aligned}$$

where $I'_j = \sigma(I_j)$ with $\sigma = \sigma_{n-2,j}$. First, note that by definition $(I'_j * \{n-1, n-2\}) \setminus n-1 = \tau_{n-1,n-2}(I'_j)$, also we have $j < k$, since $k = n - 1$. Therefore, we can deduce from the last case that $\tau_{n-1,n-2}(I'_1) = \tau_{n-1,n-2}(I'_2)$.

Finally, suppose that $\mathbb{m}_{n-1} = \mathbb{T}_{n-1,j}^-$. For (i) [8, Lemma 8] (taking $m = 0$) we have that

$$\mathbb{T}_{n-1,j}^- B_{n-1} = B_{n-2} \mathbb{T}_{n-1,j}^-$$

Therefore

$$\begin{aligned} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^- B_{n-1}) &= \vartheta_{n-1}(B_{n-2} \mathbb{T}_{n-1,j}^- E F_{I_i}) \\ &= B_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^- E F_{I_i}) \\ &= \mathbf{z} B_{n-2} \mathbb{T}_{n-2,j}^- E F_{\tau_{n-1,j}(I_i)} \end{aligned}$$

for $i = 1, 2$. Then, we have to verify that $\tau_{n-1,j}(I_1) = \tau_{n-1,j}(I_2)$. Since we are supposing $k = n - 1$, we have that

$$\begin{aligned} I_1 &= I * \{n-1, n\} \setminus n \\ I_2 &= I * \{n, j, n-1\} \setminus n \end{aligned}$$

and therefore $(I_1 * \{n-1, j\}) \setminus n-1 = (I_2 * \{n-1, j\}) \setminus n-1$. □

Lemma 10. For $n \geq 2$ and $X \in \mathcal{E}_{n-1}^{\mathbb{B}}$. We have

$$(i) \quad \vartheta_n(T_{n-1} X T_{n-1}^{-1}) = \vartheta_{n-1}(X) = \vartheta_n(T_{n-1}^{-1} X T_{n-1})$$

Proof. Consider $X = \mathbf{m}_1 \cdots \mathbf{m}_{n-2} \mathbf{m}_{n-1} E F_I$, with $I \in \mathcal{P}(\mathbf{n}_0 \setminus \{n\})$. We proceed by cases according to the value of \mathbf{m}_{n-1} .

CASE: When $\mathbf{m}_{n-1} = 1$ the results follows easily. Indeed, if $n-1 \notin \text{Supp}(I)$ the result is direct, and when $n-1 \in \text{Supp}(I)$ we have

$$\begin{aligned}
\vartheta_n(T_{n-1}XT_{n-1}^{-1}) &= \vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_I T_{n-1}^{-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{s_{n-1}(I)} \cancel{T_{n-1}^{-1}T_{n-1}}) \\
&= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{(s_{n-1}(I)) \setminus n} = \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{I \setminus n-1} = \vartheta_{n-1}(X)
\end{aligned}$$

and the right side follows analogously.

CASE: When $\mathfrak{m}_{n-1} = B_{n-1}$, we have

$$\vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_I T_{n-1}^{-1}) = \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_n EF_{s_{n-1}(I)})$$

Then when $n-1 \in \text{Supp}(I)$ we obtain

$$\begin{aligned}
\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_n EF_{s_{n-1}(I)}) &= \mathfrak{w}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{s_{n-1}(I) \setminus n} \\
&= \mathfrak{w}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}EF_{I \setminus n-1} = \vartheta_{n-1}(X)
\end{aligned}$$

the opposite case follows analogously.

CASE: When $\mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$ with $k < n-1$, we have

$$\begin{aligned}
\vartheta_n(T_{n-1}XT_{n-1}^{-1}) &= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}\mathbb{T}_{n-1,k}^\pm EF_I T_{n-1}^{-1}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,k}^\pm T_{n-1}^{-1} EF_{s_{n-1}(I)}) \\
&= \vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-2}^{-1}\mathbb{T}_{n,k}^\pm EF_{s_{n-1}(I)}) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-2}^{-1}\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(s_{n-1}(I))} \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-2,k}^\pm EF_{\tau_{n,k}(s_{n-1}(I))}
\end{aligned}$$

It is not difficult to prove that independently if $n-1$ belong to $\text{Supp}(I)$ or not, we have that $\tau_{n,k}(s_{n-1}(I)) = \tau_{n-1,k}(I)$, thus we obtain

$$\mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(I)} = \vartheta_{n-1}(X)$$

Finally we have

$$T_{n-1}XT_{n-1}^{-1} = T_{n-1}^{-1}XT_{n-1} + (\mathfrak{u} - \mathfrak{u}^{-1})(E_{n-1}XT_{n-1} - T_{n-1}XE_{n-1})$$

for $X \in \mathcal{E}_n^B$, for details see [8, Lemma 8]. Then applying this relation and (i) Lemma 9 we obtain that

$$\vartheta_n(T_{n-1}XT_{n-1}^{-1}) = \vartheta_n(T_{n-1}^{-1}XT_{n-1})$$

and (ii) follows. □

Lemma 11. For all $X \in \mathcal{E}_n^B$, we have

$$\vartheta_{n-1}(\vartheta_n(XT_{n-1})) = \vartheta_{n-1}(\vartheta_n(T_{n-1}X)) \quad (44)$$

Proof. First note that the Eq. (44) is equivalent to

$$\vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) = \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) \quad (45)$$

which can be obtained using (ii) Lemma 9 and the formula for the inverse, cf. [6]. Then, sometimes we will prove this assertion instead of (44) according to its difficulty.

As always we consider $X = \mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathfrak{m}_n EF_I$, and we will distinguish cases according to the possibilities of \mathfrak{m}_n and \mathfrak{m}_{n-1} . We omit the case $\mathfrak{m}_n = \mathfrak{m}_{n-1} = 1$ since it is straightforward.

CASE: $\mathfrak{m}_n = 1, \mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$ with $k < n-1$

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}\mathbb{T}_{n-1,k}^\pm EF_I)) \\
&= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n,k}^\pm EF_I)) \\
&= \mathbf{z}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,k}(I)}) \\
&= \mathbf{z}^2\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(\tau_{n,k}(I))}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm EF_IT_{n-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^\pm T_{n-1}EF_{s_{n-1}(I)})) \\
&= \mathbf{z}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_{\tau_{n,n-1}(s_{n-1}(I))}) \\
&= \mathbf{z}^2\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-2,k}^\pm EF_{\tau_{n-1,k}(\tau_{n,n-1}(s_{n-1}(I)))}
\end{aligned}$$

Now, note that $\tau_{n,n-1}(s_{n-1}(I)) = \tau_{n,n-1}(I)$, then, it is clear that $\tau_{n-1,k}(\tau_{n,n-1}(s_{n-1}(I))) = \tau_{n-1,k}(\tau_{n,k}(I))$.

CASE: $\mathfrak{m}_n = 1, \mathfrak{m}_{n-1} = B_{n-1}$

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-1}B_{n-1}EF_I)) \\
&= \mathbf{z}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(B_{n-1}EF_{\tau_{n,n-1}(I)})
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}EF_IT_{n-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}T_{n-1}EF_{s_{n-1}(I)})) \\
&= \mathbf{z}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\vartheta_{n-1}(B_{n-1}EF_{\tau_{n,n-1}(s_{n-1}(I))})
\end{aligned}$$

and we know by the last case that the partitions involved are equal.

CASE: $\mathfrak{m}_n = \mathbb{T}_{n,k}^\pm$ with $k < n$. In this case we will prove (45). First suppose that $n \notin \text{Supp}(I)$

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathbb{T}_{n,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}T_{n-1}\mathbb{T}_{n-1,k}^\pm EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}T_{n-1})\mathbb{T}_{n-1,k}^\pm EF_I) \quad (\text{by Lemma 10}) \\
&= \vartheta_{n-1}(\vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1})\mathbb{T}_{n-1,k}^\pm EF_I) = \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1})\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_I)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\mathbb{T}_{n,k}^\pm EF_IT_{n-1}^{-1})) \\
&= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}T_{n-1}\mathbb{T}_{n-1,k}^\pm EF_IT_{n-1}^{-1})) \\
&= \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\vartheta_n(T_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_I)T_{n-1}^{-1})) \quad (\text{by Lemma 10}) \\
&= \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1}\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_I)) = \vartheta_{n-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-1})\vartheta_{n-1}(\mathbb{T}_{n-1,k}^\pm EF_I)
\end{aligned}$$

From now on we suppose that $n \in \text{Supp}(I)$.

•SUBCASE: $\mathfrak{m}_{n-1} = 1$. First note, that for $k = n - 1$ the result follows easily. Then, we can suppose $k < n - 1$

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,k}^{\pm}EF_I)) \\
 &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n-1,k}^{\pm}EF_I)) \\
 &= \times \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^{\pm}EF_{I \setminus n}) \\
 &= \mathbf{z} \times \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-2,k}^{\pm}EF_{\tau_{n-1,k}(I \setminus n)}
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,k}^{\pm}EF_IT_{n-1}^{-1})) \\
 &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\mathbb{T}_{n,k}^{\pm}T_{n-1}^{-1}EF_{s_{n-1}(I)})) \\
 &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}T_{n-2}^{-1}\mathbb{T}_{n,k}^{\pm}EF_{s_{n-1}(I)})) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(T_{n-2}^{-1}\mathbb{T}_{n,k}^{\pm}EF_{s_{n-1}(I)}) \\
 &= \mathbf{z}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(T_{n-2}^{-1}\mathbb{T}_{n-1,k}^{\pm}EF_{\tau_{n,k}(s_{n-1}(I))}) \\
 &= \mathbf{z}\mathbf{x}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-2,k}^{\pm}EF_{\tau_{n,k}(s_{n-1}(I)) \setminus n-1}
 \end{aligned}$$

and it is easy verify that $\tau_{n,k}(s_{n-1}(I)) \setminus n - 1 = \tau_{n-1,k}(I \setminus n)$.

•SUBCASE: $\mathfrak{m}_{n-1} = B_{n-1}$. First suppose $k = n - 1$ for the negative case, that is $\mathfrak{m}_n = T_{n-1}B_{n-1}$, we have

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}T_{n-1}B_{n-1}EF_I)) \\
 &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}\cancel{T_{n-1}^{-1}}T_{n-1}B_{n-1}B_nEF_I)) \quad (\text{by (i) Lemma 3}) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1}B_nEF_I))
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}T_{n-1}B_{n-1}EF_IT_{n-1}^{-1})) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-1}B_{n-1}T_{n-1}^{-1}EF_{s_{n-1}(I)})) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1}B_nEF_{s_{n-1}(I)}))
 \end{aligned}$$

If we fix $I_1 = I$ and $I_2 = s_{n-1}(I)$,

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(B_{n-1}B_nEF_{I_i})) &= \mathbf{w}\vartheta_{n-1}(B_{n-1}EF_{I_i \setminus n}) \\
 &= \mathbf{w}\vartheta_{n-1}(B_{n-1}EF_{(I_i \setminus n) \setminus n-1}) \\
 &= \mathbf{w}^2EF_{(I_i \setminus n) \setminus n-1}
 \end{aligned}$$

the result follows easily by comparing the partitions for $i = 1, 2$. In the present case, we are supposing that $n - 1 \in \text{Supp}(I)$, for the opposite case we can proceed analogously, and we will obtain the same partitions, but this time, it will appear the parameter $\mathbf{w}\mathbf{x}$ for $i = 1, 2$ in the final result. We omit the proof for $\mathfrak{m}_n = T_{n-1}$ (positive case) since can be verified analogously.

For $k < n - 1$ we have

$$\begin{aligned}
 \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2}B_{n-1}\mathbb{T}_{n,k}^{\pm}EF_I)) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}B_{n-1}T_{n-1}T_{n-2}\mathbb{T}_{n-2,k}^{\pm}EF_I)) \\
 &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \underbrace{\vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)}))}_{\mathbf{A}} \mathbb{T}_{n-2,k}^{\pm}
 \end{aligned}$$

where $\sigma = \sigma_{n-2,k}$. Now, let's compute A

$$\begin{aligned}
A &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)})) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-1}B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)})) - (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(E_{n-1}B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)})) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-1}B_{n-1}T_{n-1}^{-1}T_{n-2}EF_{\sigma(I)})) + (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(T_{n-1}B_{n-1}E_{n-1}T_{n-2}EF_{\sigma(I)})) \\
&\quad - (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(E_{n-1}B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)})) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-2}B_nEF_{\sigma(I)})) + (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(T_{n-1}B_{n-1}T_{n-2}EF_{\sigma(I)*\{n,n-2\}})) \\
&\quad - (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(I)*\{n,n-2\}})) \\
&= \mathbf{w}\vartheta_{n-1}(T_{n-2}EF_{\sigma(I)\setminus n}) + (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(T_{n-1}T_{n-2}B_{n-2}EF_{\sigma(I)*\{n,n-2\}})) \\
&\quad - \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(B_{n-1}T_{n-2}EF_{\tau_{n,n-2}(\sigma(I)*\{n,n-2\})}) \\
&= \mathbf{wz}EF_{\tau_{n-1,n-2}(\sigma(I)\setminus n)} + \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{\tau_{n,n-2}(\sigma(I)*\{n,n-2\})}) \\
&\quad - \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{\tau_{n,n-2}(\sigma(I)*\{n,n-2\})}) \\
&= \mathbf{wz}EF_{\tau_{n-1,n-2}(\sigma(I)\setminus n)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(\mathbf{m}_1 \cdots \mathbf{m}_{n-2}B_{n-1}\mathbb{T}_{n,k}^{\pm}EF_I T_{n-1}^{-1})) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-2}^{-1}\mathbb{T}_{n,k}^{\pm}EF_J)) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\underbrace{\vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-2}^{-1}T_{n-1}T_{n-2}EF_{\sigma(J)}))}_{\mathbf{D}}\mathbb{T}_{n-2,k}^{\pm}
\end{aligned}$$

where $J = s_{n-1}(I)$ and $\sigma = \sigma_{n-2,k}$. Now, we compute D

$$\begin{aligned}
\mathbf{D} &= \vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-2}^{-1}T_{n-1}T_{n-2}EF_{\sigma(J)})) \\
&= \vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-2}T_{n-1}T_{n-2}EF_{\sigma(J)})) - (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(B_{n-1}E_{n-2}T_{n-1}T_{n-2}EF_{\sigma(J)})) \\
&= \vartheta_{n-1}(\vartheta_n(T_{n-2}B_{n-2}T_{n-1}T_{n-2}EF_{\sigma(J)})) - (\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-1}T_{n-2}EF_{\sigma(J)*\{n,n-1\}})) \\
&= \mathbf{z}\vartheta_{n-1}(T_{n-2}B_{n-2}T_{n-2}EF_{\tau_{n,n-2}(\sigma(J))}) - \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(B_{n-1}T_{n-2}EF_{\tau_{n,n-2}(\sigma(J)*\{n,n-1\})}) \\
&= \mathbf{z}\vartheta_{n-1}(B_{n-1}EF_{\tau_{n,n-2}(\sigma(J))}) + \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{(\tau_{n,n-2}(\sigma(J))*\{n-1,n-2\})}) \\
&\quad - \mathbf{z}(\mathbf{u} - \mathbf{u}^{-1})\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{\tau_{n,n-2}(\sigma(J)*\{n,n-1\})}) \\
&= \mathbf{zw}EF_{\tau_{n,n-2}(\sigma(J))\setminus n-1}
\end{aligned}$$

and it is not difficult verify that the partitions involved coincide.

•SUBCASE: $\mathbf{m}_{n-1} = \mathbb{T}_{n-1,j}^{\pm}$. We will distinguish 2 subcases

First suppose that $k < n - 1$

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}\mathbf{m}_1 \cdots \mathbf{m}_{n-2}\mathbb{T}_{n-1,j}^{\pm}\mathbb{T}_{n,k}^{\pm}EF_I)) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}T_{n-2}\mathbb{T}_{n-2,j}^{\pm}T_{n-1}\mathbb{T}_{n-1,k}^{\pm}EF_I)) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\vartheta_{n-1}(\vartheta_n(T_{n-1}T_{n-2}T_{n-2}^{-1}\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}EF_I)) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\vartheta_{n-1}(\vartheta_n(T_{n-2}T_{n-1}T_{n-2}^{-1}\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm}EF_I)) \\
&= \mathbf{m}_1 \cdots \mathbf{m}_{n-2}\vartheta_{n-1}(T_{n-2}\vartheta_n(T_{n-1}T_{n-2}^{-1}EF_{\varphi(I)})\mathbb{T}_{n-2,j}^{\pm}\mathbb{T}_{n-1,k}^{\pm})
\end{aligned}$$

where $\varphi = \sigma_{n-2,j}\sigma_{n-1,k}$. Further it is easy to prove that

$$\vartheta_n(T_{n-1}T_{n-2}^{-1}EF_{\varphi(I)}) = z T_{n-2}^{-1}EF_{\tau_{n,n-2}(\varphi(I))}$$

then we obtain the following. First we consider $j < k$.

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\cancel{T_{n-2}^{-1}T_{n-2}^{-1}} EF_{\tau_{n,n-2}(\varphi(I))} \mathbb{T}_{n-2,j}^{\pm} \mathbb{T}_{n-1,k}^{\pm}) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-2,j}^{\pm} \mathbb{T}_{n-1,k}^{\pm} EF_{\tau_{n,j}(I)}) \\ &= z^2 m_1 \cdots m_{n-2} \mathbb{T}_{n-2,j}^{\pm} \mathbb{T}_{n-2,k}^{\pm} EF_{\tau_{n-1,k}(\tau_{n,j}(I))} \end{aligned}$$

On the other hand

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(m_1 \cdots m_{n-2} \mathbb{T}_{n-1,j}^{\pm} \mathbb{T}_{n,k}^{\pm} EF_I T_{n-1}^{-1})) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,j}^{\pm} \mathbb{T}_{n,k}^{\pm} T_{n-1}^{-1} EF_J)), \quad \text{where } J = s_{n-1}(I) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} T_{n-2}^{-1} \vartheta_n(\mathbb{T}_{n,k}^{\pm} EF_J)) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} T_{n-2}^{-1} \mathbb{T}_{n-1,k}^{\pm} EF_{\tau_{n,k}(J)}) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} \mathbb{T}_{n-2,k}^{\pm} EF_{\tau_{n,k}(J)}) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} EF_{\sigma_{n-2,k}(\tau_{n,k}(J))}) \mathbb{T}_{n-2,k}^{\pm} \\ &= z^2 m_1 \cdots m_{n-2} \mathbb{T}_{n-2,j}^{\pm} EF_{\tau_{n-1,j}(\sigma_{n-2,k}(\tau_{n,k}(J)))} \mathbb{T}_{n-2,k}^{\pm} \\ &= z^2 m_1 \cdots m_{n-2} \mathbb{T}_{n-2,j}^{\pm} T_{n-2}^{-1} \mathbb{T}_{n-1,k}^{\pm} EF_{\tau_{n-1,j}(\tau_{n,k}(J))} \end{aligned}$$

and the result follows by comparing the partitions. Note that for $j \geq k$ the proof is the same but will appear the term $j+1$ instead of j , in both partitions. Finally suppose $k = n-1$, we only prove the negative case, that is $m_n = T_{n-1}B_{n-1}$, since for the positive case we can proceed analogously, and it is easier

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1} m_1 \cdots m_{n-2} \mathbb{T}_{n-1,j}^{\pm} T_{n-1} B_{n-1} EF_I)) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1} T_{n-2} \mathbb{T}_{n-2,j}^{\pm} T_{n-1} B_{n-1} EF_I)) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1}^{-1} T_{n-2} T_{n-1} \mathbb{T}_{n-2,j}^{\pm} B_{n-1} EF_I)) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-2} T_{n-1} T_{n-2}^{-1} \mathbb{T}_{n-2,j}^{\pm} B_{n-1} EF_I)) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(T_{n-2} \vartheta_n(T_{n-1} T_{n-2}^{-1} EF_{\sigma_{n-2,j}(I)}) \mathbb{T}_{n-2,j}^{\pm} B_{n-1}) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\cancel{T_{n-2}^{-1}T_{n-2}^{-1}} EF_{\tau_{n,n-2}(\sigma_{n-2,j}(I))} \mathbb{T}_{n-2,j}^{\pm} B_{n-1}) \\ &= z m_1 \cdots m_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-2,j}^{\pm} B_{n-1} EF_{\tau_{n,j}(I)}) \end{aligned}$$

now, depending if $n-1 \in \text{Supp}(I)$ or not, we can obtain

$$z m_1 \cdots m_{n-2} \mathbb{T}_{n-2,j}^{\pm} EF_{\tau_{n,j}(I) \setminus n-1} \quad \text{or} \quad z m_1 \cdots m_{n-2} \mathbb{T}_{n-2,j}^{\pm} EF_{\tau_{n,j}(I)}$$

respectively.

On the other hand

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(XT_{n-1}^{-1})) &= \vartheta_{n-1}(\vartheta_n(m_1 \cdots m_{n-2} \mathbb{T}_{n-1,j}^{\pm} T_{n-1} B_{n-1} EF_I T_{n-1}^{-1})) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,j}^{\pm} T_{n-1} B_{n-1} T_{n-1}^{-1} EF_J)) \\ &= m_1 \cdots m_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,j}^{\pm} B_{n-1} EF_J)) = C \end{aligned}$$

Now depending of $n - 1 \in \text{Supp}(I)$ or not, we obtain

$$\begin{aligned} C &= \mathbf{w}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} EF_{J \setminus n}) \quad \Bigg| \quad C = \mathbf{y}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,j}^{\pm} EF_J) \\ &= \mathbf{z}\mathbf{w}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} \mathbb{T}_{n-2,j}^{\pm} EF_{\tau_{n-1,j}(J \setminus n)} \quad \Bigg| \quad = \mathbf{z}\mathbf{y}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} \mathbb{T}_{n-2,j}^{\pm} EF_{\tau_{n-1,j}(J \setminus n)} \end{aligned}$$

respectively. And, it is easy verify that the partitions are the same, therefore this case follows.

CASE: $\mathbf{m}_n = B_n, \mathbf{m}_{n-1} = 1$

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathbf{m}_1 \cdots \mathbf{m}_{n-2} B_n EF_I T_{n-1})) \\ &= \vartheta_{n-1}(\vartheta_n(\mathbf{m}_1 \cdots \mathbf{m}_{n-2} B_n T_{n-1} EF_{s_{n-1}(I)})) \\ &= \vartheta_{n-1}(\vartheta_n(\mathbf{m}_1 \cdots \mathbf{m}_{n-2} T_{n-1} B_{n-1} EF_{s_{n-1}(I)})) \\ &= \mathbf{z}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} \vartheta_{n-1}(B_{n-1} EF_{\tau_{n,n-1}(s_{n-1}(I))}) \\ &= \mathbf{z}\mathbf{w}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} EF_{\tau_{n,n-1}(s_{n-1}(I)) \setminus n-1} \end{aligned}$$

On the other hand

$$\begin{aligned} \vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1}\mathbf{m}_1 \cdots \mathbf{m}_{n-2} B_n EF_I)) \\ &= \mathbf{m}_1 \cdots \mathbf{m}_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1} B_n EF_I)) \\ &= \mathbf{m}_1 \cdots \mathbf{m}_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1}^2 B_{n-1} T_{n-1}^{-1} EF_I)) \end{aligned}$$

expanding the square and the inverse we have that

$$\vartheta_{n-1}(\vartheta_n(T_{n-1}^2 B_{n-1} T_{n-1}^{-1} EF_I)) = A - (\mathbf{u} - \mathbf{u}^1)B + (\mathbf{u} - \mathbf{u}^1)C$$

where

$$\begin{aligned} A &:= \vartheta_{n-1}(\vartheta_n(B_{n-1} T_{n-1} EF_I)) \\ B &:= \vartheta_{n-1}(\vartheta_n(B_{n-1} E_{n-1} EF_I)) \\ C &:= \vartheta_{n-1}(\vartheta_n(E_{n-1} B_n EF_I)) \end{aligned}$$

Now, by direct computations we have that

$$\begin{aligned} A &= \mathbf{z}\vartheta_{n-1}(B_{n-1} EF_{\tau_{n,n-1}(I)}) \\ &= \mathbf{z}\mathbf{m}_1 \cdots \mathbf{m}_{n-1} EF_{\tau_{n,n-1}(I) \setminus n-1} \end{aligned}$$

$$\begin{aligned} B &= \vartheta_{n-1}(\vartheta_n(B_{n-1} EF_{I * \{n-1, n\}})) \\ &= \mathbf{x}\vartheta_{n-1}(B_{n-1} EF_{(I * \{n-1, n\}) \setminus n}) \\ &= \mathbf{x}\mathbf{w}B_{n-1} EF_{((I * \{n-1, n\}) \setminus n) \setminus n-1} \end{aligned}$$

and

$$\begin{aligned} C &= \vartheta_{n-1}(\vartheta_n(B_n EF_{I * \{n-1, n\}})) \\ &= \mathbf{w}\vartheta_{n-1}(EF_{I * \{n-1, n\} \setminus n}) \\ &= \mathbf{x}\mathbf{w}\vartheta_{n-1}(EF_{(I * \{n-1, n\} \setminus n) \setminus n-1}) \end{aligned}$$

clearly we have that $B = C$ and also that $\tau_{n,n-1}(I) \setminus n - 1 = \tau_{n,n-1}(s_{n-1}(I)) \setminus n - 1$, thus the results follows.

CASE: $\mathfrak{m}_n = B_n, \mathfrak{m}_{n-1} = B_{n-1}$

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} B_n EF_I T_{n-1})) \\ &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} B_n T_{n-1} EF_{s_{n-1}(I)})) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1} T_{n-1} B_{n-1} EF_{s_{n-1}(I)}))\end{aligned}$$

On the other hand

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-1} B_n EF_I)) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(T_{n-1} B_{n-1} B_n EF_I)) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1} T_{n-1} B_{n-1} EF_I))\end{aligned}$$

by using (i) Lemma 3. Now denote $I_1 = I$ and $I_2 = s_{n-1}(I)$. Then we have

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(B_{n-1} T_{n-1} B_{n-1} EF_{I_i})) &= \vartheta_{n-1}(B_{n-1} \vartheta_n(T_{n-1} EF_{I_i}) B_{n-1}) \\ &= z \vartheta_{n-1}(B_{n-1} EF_{\tau_{n,n-1}(I_i)} B_{n-1}) \\ &= z \vartheta_{n-1}(B_{n-1}^2 EF_{\tau_{n,n-1}(I_i)}) \\ &= z[\vartheta_{n-1}(EF_{\tau_{n,n-1}(I_i)}) + (\mathbf{v} - \mathbf{v}^{-1}) \vartheta_{n-1}(B_{n-1} EF_{\tau_{n,n-1}(I_i) * \{0, n-1\}})] \\ &= z[\mathbf{x} EF_{\tau_{n,n-1}(I_i) \setminus n-1} + \mathbf{w}(\mathbf{v} - \mathbf{v}^{-1}) \vartheta_{n-1}(B_{n-1} EF_{(\tau_{n,n-1}(I_i) * \{0, n-1\}) \setminus n-1})]\end{aligned}$$

finally it is not difficult verify that are the same for $i = 1, 2$.

CASE: $\mathfrak{m}_n = B_n, \mathfrak{m}_{n-1} = \mathbb{T}_{n-1,k}^\pm$ with $k < n-1$. We proceed first with the positive case

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^+ B_n EF_I T_{n-1})) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^+ \vartheta_n(B_n T_{n-1} EF_{s_{n-1}(I)})) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^+ \vartheta_n(T_{n-1} B_{n-1} EF_{s_{n-1}(I)})) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^+ B_{n-1} EF_{\tau_{n,n-1}(s_{n-1}(I))}) \\ &= z \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(T_{n-2} B_{n-1} EF_{\sigma(I_1)}) \mathbb{T}_{n-2,k}^+\end{aligned}$$

where $I_1 = \tau_{n,n-1}(s_{n-1}(I))$ and $\sigma = \sigma_{n-2,k}$. Now expanding the square and the inverse we obtain that

$$\begin{aligned}\vartheta_{n-1}(T_{n-2} B_{n-1} EF_{\sigma(I_1)}) &= \vartheta_{n-1}(B_{n-2} T_{n-2} EF_{\sigma(I_1)}) - (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-2} E_{n-2} EF_{\sigma(I_1)}) + \\ &\quad (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-1} E_{n-2} EF_{\sigma(I_1)}) \\ &= B_{n-2} \vartheta_{n-1}(T_{n-2} EF_{\sigma(I_1)}) - (\mathbf{u} - \mathbf{u}^{-1}) B_{n-2} \vartheta_{n-1}(EF_{\sigma(I_1) * \{n-1, n-2\}}) + \\ &\quad (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(B_{n-1} EF_{\sigma(I_1) * \{n-1, n-2\}}) \\ &= z B_{n-2} EF_{\tau_{n-1, n-2}(\sigma(I_1))} - \mathbf{x}(\mathbf{u} - \mathbf{u}^{-1}) B_{n-2} EF_{(\sigma(I_1) * \{n-1, n-2\}) \setminus n-1} + \\ &\quad \mathbf{w}(\mathbf{u} - \mathbf{u}^{-1}) EF_{(\sigma(I_1) * \{n-1, n-2\}) \setminus n-1}\end{aligned}$$

On the other hand

$$\begin{aligned}\vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^+ B_n EF_I)) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n,k}^+ B_n EF_I)) \\ &= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} [\vartheta_{n-1}(\vartheta_n(B_{n-1} T_{n-1} \mathbb{T}_{n-1,k}^+ EF_I)) - \\ &\quad (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(\vartheta_n(B_{n-1} E_{n-1} \mathbb{T}_{n-1,k}^+ EF_I)) + (\mathbf{u} - \mathbf{u}^{-1}) \vartheta_{n-1}(\vartheta_n(B_n E_{n-1} \mathbb{T}_{n-1,k}^+ EF_I))]\end{aligned}$$

Let's compute each term separately

$$\begin{aligned}
\bullet \vartheta_{n-1}(\vartheta_n(B_{n-1}T_{n-1}\mathbb{T}_{n-1,k}^+EF_I)) &= \vartheta_{n-1}(B_{n-1}\vartheta_n(T_{n-1}EF_{\varphi(I)})\mathbb{T}_{n-1,k}^+) \quad \text{with } \varphi = \sigma_{n-1,k} \\
&= \mathbf{z}\vartheta_{n-1}(B_{n-1}EF_{\tau_{n,n-1}\varphi(I)}\mathbb{T}_{n-1,k}^+) \\
&= \mathbf{z}\vartheta_{n-1}(B_{n-1}\mathbb{T}_{n-1,k}^+EF_{\tau_{n,k}(I)}) \quad (\text{by (iii) Lemma 5}) \\
&= \mathbf{z}\vartheta_{n-1}(T_{n-2}B_{n-2}\mathbb{T}_{n-2,k}^+EF_{\tau_{n,k}(I)}) \\
&= \mathbf{z}\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{\sigma(\tau_{n,k}(I))})\mathbb{T}_{n-2,k}^+ \\
&= \mathbf{z}^2B_{n-2}EF_{\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))}\mathbb{T}_{n-2,k}^+
\end{aligned}$$

$$\begin{aligned}
\bullet \vartheta_{n-1}(\vartheta_n(B_{n-1}E_{n-1}\mathbb{T}_{n-1,k}^+EF_I)) &= \vartheta_{n-1}(\vartheta_n(B_{n-1}\mathbb{T}_{n-1,k}^+E_{n,k}EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(B_{n-1}\mathbb{T}_{n-1,k}^+EF_{I*\{n,k\}})) \\
&= \mathbf{x}\vartheta_{n-1}(B_{n-1}\mathbb{T}_{n-1,k}^+EF_{(I*\{n,k\})\setminus n}) \\
&= \mathbf{x}\vartheta_{n-1}(T_{n-2}B_{n-2}EF_{\sigma(\tau_{n,k}(I))})\mathbb{T}_{n-2,k}^+ \\
&= \mathbf{z}\mathbf{x}B_{n-2}EF_{\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))}\mathbb{T}_{n-2,k}^+
\end{aligned}$$

$$\begin{aligned}
\bullet \vartheta_{n-1}(\vartheta_n(B_nE_{n-1}\mathbb{T}_{n-1,k}^+EF_I)) &= \vartheta_{n-1}(\vartheta_n(B_n\mathbb{T}_{n-1,k}^+E_{n,k}EF_I)) \\
&= \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,k}^+B_nEF_{I*\{n,k\}})) \\
&= \mathbf{w}\vartheta_{n-1}(\mathbb{T}_{n-1,k}^+EF_{(I*\{n,k\})\setminus n}) \\
&= \mathbf{z}\mathbf{w}\mathbb{T}_{n-2,k}^+EF_{\tau_{n-1,k}(\tau_{n,k}(I))} \\
&= \mathbf{z}\mathbf{w}EF_{\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))}\mathbb{T}_{n-2,k}^+
\end{aligned}$$

Finally, by using (iii) Lemma 5 we have

$$\begin{aligned}
\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I))) &= \tau_{n-1,n-2}((\tau_{n,n-2}(\sigma(I)))) \\
&= (\sigma(I) * \{n, n-1, n-2\}) \setminus \{n, n-1\}
\end{aligned}$$

and, on the other hand

$$\begin{aligned}
\tau_{n-1,n-2}(\sigma(I_1)) &= \tau_{n-1,n-2}(\tau_{n,n-1}(\sigma(s_{n-1}(I)))) \\
&= \tau_{n-1,n-2}(\tau_{n,n-1}(s_{n-1}(\sigma(I)))) \\
&= (s_{n-1}(\sigma(I)) * \{n, n-1, n-2\}) \setminus \{n, n-1\}
\end{aligned}$$

since s_{n-1} only moves the elements n and $n-1$, and these are removed by the partition, the equality follows.

For the negative case, we have

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(XT_{n-1})) &= \vartheta_{n-1}(\vartheta_n(\mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^- B_n T_{n-1} E F_{s_{n-1}(I)})) \\
&= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n-1,k}^- T_{n-1} B_{n-1} E F_{s_{n-1}(I)})) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^- B_{n-1} E F_{\tau_{n,n-1}(s_{n-1}(I))}) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(B_{n-2} \mathbb{T}_{n-1,k}^- E F_{\tau_{n,n-1}(s_{n-1}(I))}) \quad (\text{by (ii) Lemma 3}) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-2} \vartheta_{n-1}(\mathbb{T}_{n-1,k}^- E F_{\tau_{n,n-1}(s_{n-1}(I))}) \\
&= \mathfrak{z}^2 \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-2} \mathbb{T}_{n-2,k}^- E F_{\tau_{n-1,k}(\tau_{n,n-1}(s_{n-1}(I)))} \\
&= \mathfrak{z}^2 \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-2} E F_{\sigma(\tau_{n-1,k}(\tau_{n,n-1}(s_{n-1}(I))))} \mathbb{T}_{n-2,k}^-
\end{aligned}$$

and for the other side

$$\begin{aligned}
\vartheta_{n-1}(\vartheta_n(T_{n-1}X)) &= \vartheta_{n-1}(\vartheta_n(T_{n-1} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \mathbb{T}_{n-1,k}^- B_n E F_I)) \\
&= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(\mathbb{T}_{n,k}^- B_n E F_I)) \\
&= \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(\vartheta_n(B_{n-1} \mathbb{T}_{n,k}^- E F_I)) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(B_{n-1} \mathbb{T}_{n-1,k}^- E F_{\tau_{n,k}(I)}) \\
&= \mathfrak{z} \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \vartheta_{n-1}(T_{n-2} B_{n-2} E F_{\sigma(\tau_{n,k}(I))}) \mathbb{T}_{n-2,k}^- \\
&= \mathfrak{z}^2 \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-2} E F_{\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))} \mathbb{T}_{n-2,k}^- \\
&= \mathfrak{z}^2 \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} B_{n-2} E F_{\tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))} \mathbb{T}_{n-2,k}^-
\end{aligned}$$

Finally, using (iii) Lemma 5 we have that

$$\begin{aligned}
\sigma(\tau_{n-1,k}(\tau_{n,n-1}(s_{n-1}(I)))) &= \tau_{n-1,n-2}(\tau_{n,n-1}(\sigma(s_{n-1}(I)))) \\
&= [(\sigma(s_{n-1}(I)) * \{n, n-1\}) \setminus n] * \{n-1, n-2\} \setminus n-1 \\
&= [(\sigma(s_{n-1}(I)) * \{n, n-1, n-2\}) \setminus \{n, n-1\}] \\
&= [(\sigma(I) * \{n, n-1, n-2\}) \setminus \{n, n-1\}] \\
&= \tau_{n-1,n-2}(\tau_{n,n-2}(\sigma(I))) = \tau_{n-1,n-2}(\sigma(\tau_{n,k}(I)))
\end{aligned}$$

□

Let be $\text{tr}_n : \mathcal{E}_n^B \rightarrow \mathbb{L}$ the linear map defined inductively as follows by, $\text{tr}_1 = \vartheta_1$ and

$$\text{tr}_n := \text{tr}_{n-1} \circ \vartheta_n \quad (46)$$

and, let us denote by tr the family $\{\text{tr}_n\}_{n \geq 1}$. Then, we have the following result

Theorem 3. tr is a Markov trace on $\{\mathcal{E}_n^B\}_{n \geq 1}$. That is, for every $n \geq 1$ the linear map $\text{tr}_n : \mathcal{E}_n^B \rightarrow \mathbb{L}$ satisfies the following properties.

- i) $\text{tr}_n(1) = 1$
- ii) $\text{tr}_{n+1}(X T_n) = \text{tr}_{n+1}(X E_n T_n) = \text{ztr}_n(X)$
- iii) $\text{tr}_{n+1}(X E_n) = \text{xtr}_n(X)$
- iv) $\text{tr}_{n+1}(X B_n) = \text{ytr}_n(X)$
- v) $\text{tr}_{n+1}(X B_n E_n) = \text{tr}_{n+1}(X B_n F_{n+1}) = \text{wtr}_n(X)$
- vi) $\text{tr}_n(XY) = \text{tr}_n(YX)$ where $X, Y \in \mathcal{E}_n^B$

for all $n \geq 1$.

Proof. Rules (ii)–(v) are direct consequences of Lemma 8 (ii). We will prove rule (vi) by induction on n . For $n = 1$, the rule holds since \mathcal{E}_1^B is commutative. Suppose now that (vi) is true for all k less than n . For $Y \in$

\mathcal{E}_{n-1}^B and $X \in \mathcal{E}_n^B$ the result follows easily by Lemma 8 and induction hypothesis, cf. [2, Theorem 3]. Thus, $\text{tr}_n(XY) = \text{tr}_n(YX)$ for all $X \in \mathcal{E}_n^B$ and $Y \in \mathcal{E}_{n-1}^B$.

Further, for $Y \in \{T_{n-1}, E_{n-1}\}$ we have

$$\text{tr}_n(XY) = \text{tr}_{n-2}(\text{tr}_{n-1}(\text{tr}_n(XY))) = \text{tr}_{n-2}(\text{tr}_{n-1}(\text{tr}_n(YX)))$$

by using Lemmas 11 and 9.

Therefore, we have

$$\text{tr}_n(XY) = \text{tr}_n(XY)$$

for all $X \in \mathcal{E}_n^B$ and $Y \in \mathcal{E}_{n-1}^B \cup \{T_{n-1}, E_{n-1}\}$, thus, having in mind the linearity of tr_n , the result follows. \square

5.2. Knot invariants from \mathcal{E}_n^B . In order to define a new invariant of classical links in the solid torus, we recall some necessary facts. The closure of a braid α in the group \widetilde{W}_n (recall Section 2.1), is defined by joining with simple (unknotted and unlinked) arcs its corresponding endpoints, and it is denoted by $\widehat{\alpha}$. The result of closure, $\widehat{\alpha}$, is a link in the solid torus, denoted ST . This can be regarded by viewing the closure of the fixed strand as the complementary solid torus. For an example of a link in the solid torus see Figure 5. By the analogue of the Markov theorem for ST (cf. for example [13, 14]), isotopy classes of oriented links in ST are in bijection with equivalence classes of $\bigcup_n \widetilde{W}_n$, the inductive limit of braid groups of type B, respect to the equivalence relation \sim_B :

- (i) $\alpha\beta \sim_B \beta\alpha$
- (ii) $\alpha \sim_B \alpha\sigma_n$ and $\alpha \sim_B \alpha\sigma_n^{-1}$

for all $\alpha, \beta \in W_n$.

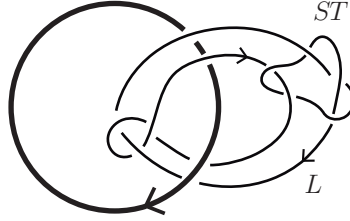


FIGURE 5. A link in the solid torus.

We set

$$L := \frac{z - (u - u^{-1})x}{z} \quad \text{and} \quad D := \frac{1}{z\sqrt{L}}. \quad (47)$$

And, let us denote π_L the representation of \widetilde{W}_n in \mathcal{E}_n^B , given by $\sigma_i \mapsto \sqrt{L}T_i$ and $\rho_1 \mapsto B_1$. Then, for $\alpha \in \widetilde{W}_n$, we define

$$\overline{\Delta}_B(\alpha) := (D)^{n-1}(\text{tr}_n \circ \pi_L)(\alpha) \quad (48)$$

it is well know that the previous expression can be rewritten as follows

$$\overline{\Delta}_B(\alpha) = (D)^{n-1}(\sqrt{L})^{e(\alpha)}(\text{tr}_n \circ \pi)(\alpha) \quad (49)$$

where $e(\alpha)$ is the exponent sum of the σ_i 's appearing in the braid α , and π is the natural representation of \widetilde{W}_n in \mathcal{E}_n^B .

Theorem 4. *Let L be a link in ST obtained by closure a braid $\alpha \in \widetilde{W}_n$. Then the map $L \mapsto \overline{\Delta}_B(\alpha)$ defines an isotopy invariant of links in ST .*

Proof. The proof follows by using the Markov trace properties, and by the definition of the normalization element L . \square

Remark 7. Note that the classical links can be regarded as a link in ST , in fact, a classical link can be obtained by closure a braid $\alpha \in \widetilde{W}_n$ whose doesn't contain ρ_1 in its expression. Thus, the invariant $\overline{\Delta}_B$ restricted to classical links coincide with the invariant $\overline{\Delta}$ given in [2, Section 6], and therefore it's more powerful than the Homflypt polynomial in that case.

Remark 8. The Markov trace tr from Theorem 3 was constructed with the aim to define invariants for “tied links in the solid torus”, having as reference [3]. However, to do that, it is necessary to introduce these new objects from the beginning, which is a problem itself. Then, we will study this subject in a future work (in progress).

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